# Cylindrically Symmetric Marder Universe and Its Proper Teleparallel Homothetic Motions 

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#### Abstract

This paper is concerned to find proper teleparallel homothetic vector fields in cylindrically symmetric Marder universe. Our method comprised of some algebraic and direct integration technique. It is shown that in the absence of curvature and in the presence of torsion the above spacetime admits one proper teleparallel homothetic vector field for a particular choice of the metric functions. KEYWORDS: Torsion, Direct integration technique, Teleparallel Lie derivative, Weitzenböck connections, Tetrad fields, Teleparallel homothetic vector fields.


## INTRODUCTION

The role of symmetries in general relativity is obvious for different purposes. In relativity theory many authors have studied different symmetries of well known spacetimes like Killing, homothetic, conformal and self-similar vector fields [1-10]. These symmetries were studied in the presence of curvature in the spacetimes. There are other theories of gravitation which do not consider curvature in the spacetime and torsion is considered responsible for gravitational interaction [11]. Symmetries in the presence of torsion were ignored to study until the discovery of teleparallel Lie derivative by M. Sharif and M. J. Amir [12]. This new definition of Lie derivative in the presence of torsion opened a door for the study of different symmetries. In the pioneer paper [12] the authors explored teleparallel Killing vectors for Einstein Universe. Later on this study was carried out on more complicated spacetimes and different well known spacetimes have been classified according to their Killing [13,14,15,16,17,18], homothetic[19,20,21,22,23] and conformal vector fields [24,25]. Keeping in mind the extensive use of symmetries in different geometrical and physical perspectives of spacetime, we are interested to study proper homothetic vector fields in cylindrically symmetric Marder Universe.
TELEPARALLEL THEORY: (Some Basics)
In this section a brief description of teleparallel theory is given. In general relativity theory a covariant derivative is needed instead of partial derivative which is coordinate dependent. This covariant derivative for second rank tensor in teleparallel theory is defined as [11]

$$
\begin{equation*}
\nabla_{\rho} B_{\mu \nu}=B_{\mu v, \rho}-\Gamma_{\rho v}^{\theta} B_{\mu \theta}-\Gamma_{\mu \rho}^{\theta} B_{v \theta}, \tag{1}
\end{equation*}
$$

where comma stands for partial derivative and $\Gamma^{\theta}{ }_{\rho v}$ represents Weitzenböck connections defined by [11]

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\theta}=O_{a}^{\theta} \partial_{\nu} O_{\mu}^{a}, \tag{2}
\end{equation*}
$$

where $O_{a}{ }^{v}$ is a non-trivial tetrad field and $O^{a} \mu$ represents inverse field. This tetrad field satisfies a relation

$$
\begin{equation*}
O^{a}{ }_{\mu} O_{a}{ }^{v}=\delta_{\mu}{ }^{v}, \quad O_{\mu}^{a} O_{b}{ }^{\mu}=\delta_{b}{ }^{a} . \tag{3}
\end{equation*}
$$

The Riemannian metric can be generated through the above tetrad field and its inverse as

$$
\begin{equation*}
g_{\mu \nu}=\eta_{a b} O^{a}{ }_{\mu} O_{\nu}^{b}, \tag{4}
\end{equation*}
$$

[^0]where $\eta_{a b}=\operatorname{diag}(-1,1,1,1)$ is the Minkowski metric. Torsion components are basically the difference of Weitzenböck connections i.e.
\[

$$
\begin{equation*}
T^{\theta}{ }_{\mu \nu}=\Gamma^{\theta}{ }_{\nu \mu}-\Gamma^{\theta}{ }_{\mu \nu} . \tag{5}
\end{equation*}
$$

\]

Torsion is anti symmetric in lower indices. In the presence of torsion in a spacetime, teleparallel Killing equation is defined as [12]
$L_{X}^{T} g_{\mu \nu}=g_{\mu \nu, \rho} X^{\rho}+g_{\rho \nu} X^{\rho}{ }_{, \mu}+g_{\mu \rho} X^{\rho}{ }_{, \nu}+X^{\rho}\left(g_{\theta \nu} T^{\theta}{ }_{\mu \rho}+g_{\mu \theta} T^{\theta}{ }_{\nu \rho}\right)=0$.
In order to find proper teleparallel homothetic motions in Marder universe, we need to use the definition (6) in its extended form as
$L_{X}^{T} g_{\mu \nu}=g_{\mu \nu, \rho} X^{\rho}+g_{\rho \nu} X^{\rho}{ }_{, \mu}+g_{\mu \rho} X^{\rho}{ }_{, \nu}+X^{\rho}\left(g_{\theta \nu} T^{\theta}{ }_{\mu \rho}+g_{\mu \theta} T^{\theta}{ }_{\nu \rho}\right)=2 \phi g_{\mu \nu}$.
where $\phi=$ constant and when $\phi=0$, equation (7) will reduce to equation (6).

## MAIN RESULTS:

The line element for cylindrically symmetric Marder universe in the usual coordinate system is given by

$$
\begin{equation*}
d s^{2}=S^{2}(t)\left[-d t^{2}+d x^{2}\right]+D^{2}(t) d y^{2}+F^{2}(t) d z^{2} \tag{8}
\end{equation*}
$$

where $S, D$ and $F$ are no-where zero functions of $t$ only. Also the components of tetrad and its inverse are obtained by using the relation (4) as

$$
O^{a}{ }_{\mu}=\left(\begin{array}{cccc}
\sqrt{S(t)} & 0 & 0 & 0  \tag{9}\\
0 & \sqrt{S(t)} & 0 & 0 \\
0 & 0 & \sqrt{D(t)} & 0 \\
0 & 0 & 0 & \sqrt{F(t)}
\end{array}\right), \quad O_{a}^{\mu}=\left(\begin{array}{cccc}
\frac{1}{\sqrt{s(t)}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{S(t)}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{D(t)}} & 0 \\
& & 0 & \frac{1}{\sqrt{F(t)}}
\end{array}\right) .
$$

The non vanishing torsion components by using (5) are

$$
\begin{equation*}
T_{10}^{1}=-T_{10}^{1}=-\frac{S^{\prime}}{S}, T_{20}^{2}=-T_{02}^{2}=-\frac{D^{\prime}}{D}, T_{30}^{3}=-T_{03}^{3}=-\frac{F^{\prime}}{F} \tag{10}
\end{equation*}
$$

where prime stands for derivative with respect to $t$. Now using (8) and (10) in (6) we get the teleparallel Killing equations as follows:

$$
\begin{align*}
& S^{\prime}(t) X^{0}+S(t) X_{, 0}^{0}=S(t) \phi  \tag{11}\\
& S^{\prime}(t) X^{1}-S(t) X_{, 0}^{1}+S(t) X_{, 0}^{0}=0  \tag{12}\\
& \frac{S^{2}}{D^{2}} X_{, 2}^{0}-X_{, 0}^{2}-\frac{D^{\prime}}{D} X^{2}=0  \tag{13}\\
& \frac{S^{2}}{F^{2}} X_{, 3}^{0}-X_{, 0}^{3}-\frac{F^{\prime}}{F} X^{3}=0  \tag{14}\\
& X_{, 1}^{1}=X_{, 2}^{2}=X_{, 3}^{3}=\phi  \tag{15}\\
& \frac{S^{2}}{D^{2}} X_{, 2}^{1}+X_{, 1}^{2}=0 \tag{16}
\end{align*}
$$

$$
\begin{align*}
& \frac{S^{2}}{F^{2}} X_{, 3}^{1}-X_{, 1}^{3}=0  \tag{17}\\
& \frac{D^{2}}{F^{2}} X_{, 3}^{2}-X_{, 2}^{3}=0 \tag{18}
\end{align*}
$$

Now integrating equations (11) and (15) we get

$$
\begin{array}{ll}
X^{0}=\phi \frac{1}{S} \int S d t+\frac{1}{S} E^{1}(x, y, z), & X^{1}=\phi x+E^{2}(t, y, z)  \tag{19}\\
X^{2}=\phi y+E^{3}(t, x, z), & X^{3}=\phi z+E^{4}(t, x, y)
\end{array}
$$

where $E^{1}(x, y, z), E^{2}(t, y, z), E^{3}(t, x, z)$ and $E^{4}(t, x, y)$ are functions of integration which are to be determined. It is important to remind the readers that when $S=$ constant, Marder universe becomes a special case of Bianchi type I spacetimes and the results of which can be seen in [13]. Here we are interested to obtain teleparallel proper homothetic vector field only for the case when $S \neq$ constant. Result is written directly here and lengthy details are omitted. We obtained the teleparallel homothetic vector fields as follows:

$$
\begin{align*}
& X^{0}=\phi \frac{1}{S} \int S d t+\frac{1}{S}\left[\frac{x^{2}}{2} c_{1}+x c_{6}+y c_{11}+z c_{14}+c_{15}\right], \quad X^{1}=\phi x+\frac{t}{S} c_{6}+\frac{1}{S} c_{8} \\
& X^{2}=\phi y+z c_{12}+c_{11} \int S^{2} d t+c_{13}, \quad X^{3}=\phi z-y c_{12}+c_{14} \int S^{2} d t+c_{16} \tag{20}
\end{align*}
$$

where $c_{1}, c_{6}, c_{8}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16} \in \mathfrak{R}$. The line element for cylindrically symmetric Marder universe after a suitable rescaling of $y$ and $z$ takes the form

$$
\begin{equation*}
d s^{2}=\left(\frac{c_{1}}{\phi} t+c_{2}\right)^{2}\left[-d t^{2}+d x^{2}\right]+d y^{2}+d z^{2}, \tag{21}
\end{equation*}
$$

where $c_{1}, c_{2} \in \mathfrak{R}\left(c_{1} \neq 0\right)$. The above spacetime (21) admits nine linearly independent teleparallel homothetic vector fields in which eight are teleparallel Killing vector fields given as $\frac{1}{S(t)} \frac{\partial}{\partial t}$, $\frac{1}{S(t)} \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}, \quad \frac{1}{S(t)}\left(x \frac{\partial}{\partial t}+t \frac{\partial}{\partial x}\right), \quad \frac{1}{S(t)} y \frac{\partial}{\partial t}+G(t) \frac{\partial}{\partial y}, \quad \frac{1}{S(t)} z \frac{\partial}{\partial t}+G(t) \frac{\partial}{\partial z} \quad$ and $z \frac{\partial}{\partial y}-y \frac{\partial}{\partial z}$, where $G(t)=\int S^{2}(t) d t, \quad$ and one is proper teleparallel homothetic vector field. Proper teleparallel homothetic vector field after subtracting teleparallel Killing vector fields from (20) is given as

$$
\begin{equation*}
X^{0}=\frac{\phi}{S(t)} \int S(t) d t+\frac{1}{S(t)} \frac{x^{2}}{2} c_{1}, \quad X^{1}=\phi x, \quad X^{2}=\phi y, \quad X^{3}=\phi z \tag{22}
\end{equation*}
$$

## CONCLUSION

In this paper we explored proper teleparallel homothetic vector fields in cylindrically symmetric Marder universe. In order to find proper homothetic vector fields in context of teleparallel theory we applied some algebraic and direct integration technique. This study is carried out in the presence of torsion and it comes out that Marder universe admits one proper teleparallel homothetic vector field for a particular choice of the metric functions. The metric which admit proper homothety is given in equation (21).

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