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Multigrid Method Based on Transformation Free Higher Order Scheme for Solving 3D Helmholtz Equation on Nonuniform Grids

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ABSTRACT

Higher-order compact difference schemes can achieve higher order accuracy on uniform grids. However, in some cases these may not achieve the desired accuracy. Therefore, this paper proposes a multigrid method based on higher-order compact difference scheme for 3D Helmholtz equation on nonuniform grids. Interpolation and restriction operators are designed accordingly. The suggested scheme has up to third to fourth order accuracy. Lastly, some numerical examples are presented in order to show the accuracy and performance of the considered scheme.
KEYWORDS: Helmholtz equation, Compact iterative schemes, Multigrid method, Non uniform grids.

1. INTRODUCTION

Two and three dimensional elliptic PDEs play a pivotal role in different fields of science and technology. Higher order compact schemes (HOC) are used for the solution of Helmholtz equation and other elliptic PDEs [3, 24]. The three-dimensional (3D) Helmholtz equation is consider

$$u_{xx} + u_{yy} + u_{zz} + \ell^2 u = f(x, y, z), (x, y, z) \in \Omega, \quad (1)$$

where Ω is a cubic domain and k is a wave number. The forcing function $f(x, y, z)$ and the solution $u(x, y, z)$ have the required continuous differentiability up to a specific order. Helmholtz equation has many real world applications like elasticity, electromagnetic waves, acoustic wave scattering, weather and climate prediction, water wave propagation, noise reduction in silencers and radar scattering. In this paper, we use a finite difference approximation on nonuniform grids in discrete domain to obtain a scheme up to fourth order accuracy. We also considered Helmholtz equation with constant value of k. The discretized form of equation (1) is

$$\mathrm{d}_x^2 u_{i,j,k} + \mathrm{d}_y^2 u_{i,j,k} + \mathrm{d}_z^2 u_{i,j,k} + \ell^2 u_{i,j,k} = f_{i,j,k} + O(h^2). \quad (2)$$

Equation (1) has been solved by different techniques such as finite-difference method (FDM) [22], fast Fourier transform- based (FFT) methods [15], finite element method (FEM) [14], the spectral-element method [18], compact finite-difference method [19] and multigrid methods [1]. Multigrid method based on HOC schemes is among the most efficient iterative technique for solving PDEs [12, 30].

In FDM the number of mesh points will be enlarged to increase the accuracy however, it will also increase the computational time. Helmholtz equation is solved by FEM and spectral-element method, but the limitations of these methods are of high computational cost [18]. Many iterative techniques for Helmholtz equation suffer due to their slow convergence. The investigation for fast iterative methods to achieve higher order accuracy for PDEs is more attractive a area of research.

Multigird method together with the HOC schemes on uniform grids are developed in [9,10 ,11 ,30]. In most cases, when sudden changes occur in a flow, the step sizes have to be rectified over the entire domain. Under these situations, where points are concentrated in the regions of sharp variation local mesh refinement procedures [3, 1, 4, 16, 25, 32] are necessary, thus dramatically reducing the computational time and computer storage. Cao and Ge developed a multigrid method with HOC scheme on nonuniform grids for solving 2D convection diffusion equation [4]. This paper is based on approach that an interpolation operator and a projection operator that are suited for HOC scheme using nonuniform mesh is represented by transformation-free HOC scheme on nonuniform grids. The main focus in this paper is to develop multigrid method based on HOC scheme on nonuniform grids for solving of 3D Helmholtz equation. To the best of our knowledge 3D Helmholtz equation is not solved by multigrid method based on HOC scheme on nonuniform grids.

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2 Mathematical formulation

Consider a cubic domain $(x,y,z)\in[a_1,a_2]\times[b_1,b_2]\times[c_1,c_2]$. Discretization is performed on a three-dimensional nonuniform girds points. The above intervals $[a_1,a_2],[b_1,b_2]$ and $[c_1,c_2]$ are divide into subintervals

$$a_1=x_0,x_1,x_2,\ldots,x_{N_x}=a_2,\qquad b_1=y_0,y_1,y_2,\ldots,y_{N_y}=b_2\qquad and\qquad c_1=z_0,z_1,z_2,\ldots,z_{N_z}=c_2.$$

In x-direction, consider $h_x=\dfrac{a_2-a_1}{N_x}$, the forward and backward step sizes are gives by)

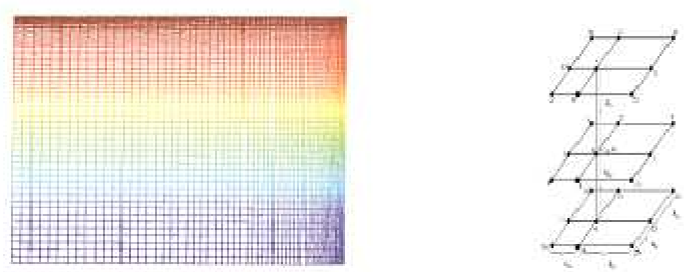
$$h_{fx}=x_{i+1}-x_i=\theta_{fx}h_x,\ h_{bx}=x_i-x_{i-1}=\theta_{bx}h_x,\ 1\le i<N_x-1.$$

Also for y-direction, $h_y=\dfrac{b_2-b_1}{N_y}$,

$$h_{fy}=y_{j+1}-y_j=\theta_{fy}h_y,\ h_{by}=y_j-y_{j-1}=\theta_{by}h_y,\ 1\le j<N_y-1.$$

Furthermore, we define, $\alpha_x,\beta_x,\gamma_x$ as $\alpha_x=\theta_{fx}\theta_{bx}$, $\beta_x=\theta_{fx}+\theta_{bx}$ and $\gamma_x=\theta_{fx}-\theta_{fx}$.

Similarly, α_y , β_y , γ_y and α_z , β_z , γ_z can be find accordingly for y and z directions. If $\theta_{fx}=\theta_{bx}=1(h_{fx}=h_{bx},h_{fy}=h_{by},h_{fz}=h_{bz})$ then the grids turn to be uniform. The approximate value of a function $u(x, y, z)$ at the interior grids points (x_i,y_j,z_k) is represented by u_0 and the approximate values of other twenty six (26) nearest grid points are determined by u_i , i=1,2,3,...,26, as in Fig. 1(b).



(a) (b)

Figure 1: (a) Non uniform grids distribution in xy-plane. (b) Stencil of nonuniform 3D grids.

Taylor series expansion is performed for appropriate description of sufficient smooth function $u(x,y,z)$ in the given domain at points 1 and 3 are

$$u_1=u_0+\theta_{fx}h_x\partial_x u_0+\frac{\theta_{fx}^2h_x^2}{2}\partial_x^2u_0+\frac{\theta_{fx}^3h_x^3}{6}\partial_x^3u_0+\frac{\theta_{fx}^4h_x^4}{24}\partial_x^4u_0+\frac{\theta_{fx}^5h_x^5}{120}\partial_x^5u_0+O(\theta_{fx}^6h_x^6),\tag{3}$$

$$u_3=u_0-\theta_{bx}h_x\partial_x u_0+\frac{\theta_{bx}^2h_x^2}{2}\partial_x^2u_0-\frac{\theta_{bx}^3h_x^3}{6}\partial_x^3u_0+\frac{\theta_{bx}^4h_x^4}{24}\partial_x^4u_0-\frac{\theta_{bx}^5h_x^5}{120}\partial_x^5u_0+O(\theta_{bx}^6h_x^6).\tag{4}$$

Multiplying equation (3) by θ_{bx} and equation (4) by θ_{fx}, then adding and solving for the second derivative which gives

$$\partial_x^2u_0=\frac{2}{\alpha_x\beta_x h_x^2}(\theta_{bx}u_1-\beta_xu_0+\theta_{fx}u_3)-\frac{h_x}{3}\gamma_x\partial_x^3u_0-\frac{h_x^2}{12}(\beta_x^2-3\alpha_x)\partial_x^4u_0-\frac{h_x^3}{60}(\beta_x^2-2\alpha_x)\gamma_x\partial_x^5u_0+O(\eta_xh_x^4),\tag{5}$$

in equation (5) $\eta_x=\beta_x^4-5\alpha_x(\beta_x^2-\alpha_x)$, hence the second order central difference operator along x– direction is defined as

$$\delta_x^2u_0=\frac{2}{\alpha_x\beta_xh_x^2}(\theta_{bx}u_1-\beta_xu_0+\theta_{fx}u_3),\tag{6}$$

if $\theta_{fx} = \theta_{bx} = 1$, equation (6) reduces to uniform grids of central difference operator. Therefore, the second order derivative for x direction is

$$\partial_x^2 u_0 = \delta_x^2 u_0 - \frac{h_x}{3}\gamma_x \partial_x^3 u_0 - \frac{h_x^2}{12}(\beta_x^2 - 3\alpha_x)\partial_x^4 u_0 - \frac{h_x^3}{60}(\beta_x^2 - 2\alpha_x)\gamma_x \partial_x^5 u_0 + O(\eta_x h_x^4) \tag{7}$$

The approximation of the second order derivative for the variables y and z can be finding accordingly. Therefore, the central difference scheme for Helmholtz equation can be discretized as

$$\delta_x^2 u_0 + \delta_y^2 u_0 + \delta_z^2 u_0 + \ell^2(u_0) = f_0 + \tau_0. \tag{8}$$

where τ_0 is the truncation error and is defined as

$$\tau_0 = H_1 \frac{\partial^3 u_0}{\partial x^3} + K_1 \frac{\partial^3 u_0}{\partial y^3} + L_1 \frac{\partial^3 u_0}{\partial z^3} + H_2 \frac{\partial^4 u_0}{\partial x^4} + K_2 \frac{\partial^4 u_0}{\partial y^4} + L_2 \frac{\partial^4 u_0}{\partial z^4} + H_3 \frac{\partial^5 u_0}{\partial x^5} + K_3 \frac{\partial^5 u_0}{\partial y^5} + L_2 \frac{\partial^5 u_0}{\partial z^5} + O(\eta_x h_x^4) + O(\eta_y h_y^4) + O(\eta_x z h_z^4). \tag{9}$$

Here H_1, H_2, H_3, L_1, L_2, L_3 and K_1, K_2, K_3 are defined as

$$H_1 = \frac{1}{3}h_x\gamma_x, \quad K_1 = \frac{1}{3}h_y\gamma_y, \quad L_1 = \frac{1}{3}h_z\gamma_z, \quad H_2 = \frac{1}{12}h_x^2(\beta_x^2 - 3\alpha_x), \quad K_2 = \frac{1}{12}h_y^2(\beta_y^2 - 3\alpha_y),$$

$$K_3 = \frac{1}{60}h_y^3(\beta_y^2 - 2\alpha_y)\gamma_y, \; L_2 = \frac{1}{12}h_z^2(\beta_z^2 - 3\alpha_z), \; H_3 = \frac{1}{60}h_x^3(\beta_x^2 - 2\alpha_x)\gamma_x,$$

$$L_3 = \frac{1}{60}h_z^3(\beta_z^2 - 2\alpha_z)\gamma_z, \; \eta_y = \beta_y^4 - 5\alpha_y(\beta_y^2 - \alpha_y), \; \eta_z = \beta_z^4 - 5\alpha_z(\beta_z^2 - \alpha_z).$$

If τ_0 is dropp off from equation (8), the central difference scheme (CDS) for nonuniform grids will be

$$\delta_x^2 u_0 + \delta_y^2 u_0 + \delta_z^2 u_0 + \ell^2(u_0) = f_0. \tag{10}$$

According to the definition of δ_x^2, δ_y^2 and δ_z^2 the central difference scheme can be written as

$$-2\left(\frac{1}{\alpha_x h_x^2} + \frac{1}{\alpha_y h_y^2} + \frac{1}{\alpha_z h_z^2}\right)u_0 + \frac{2\theta_{bx}u_1}{\alpha_x\beta_x h_x^2} + \frac{2\theta_{by}u_2}{\alpha_y\beta_y h_y^2} + \frac{2\theta_{fx}u_3}{\alpha_x\beta_x h_x^2} + \frac{2\theta_{fy}u_4}{\alpha_y\beta_y h_y^2} + \frac{2\theta_{bz}u_5}{\alpha_z\beta_z h_z^2} + \frac{2\theta_{fz}u_6}{\alpha_z\beta_z h_z^2} = f_0. \tag{11}$$

In above equation (11) only seven grids points are involved. From the definition of τ_0, it is observed that when $h_{fx}=h_{bx}$, $h_{fy}=h_{by}$ and $h_{fz}=h_{bz}$, then equation (11) is of third order accuracy. In order to improve the order of accuracy we consider

$$H_1\frac{\partial^3u_0}{\partial x^3}+K_1\frac{\partial^3u_0}{\partial y^3}+L_1\frac{\partial^3u_0}{\partial z^3}=\left(H_1\frac{\partial}{\partial x}+K_1\frac{\partial}{\partial y}+L_1\frac{\partial}{\partial z}\right)\left(\frac{\partial^2u_0}{\partial x^2}+\frac{\partial^2u_0}{\partial y^2}+\frac{\partial^2u_0}{\partial z^2}\right)-\left(H_1\frac{\partial^3u_0}{\partial x\partial y^2}+H_1\frac{\partial^3u_0}{\partial x\partial z^2}+K_1\frac{\partial^3u_0}{\partial y\partial x^2}+K_1\frac{\partial^3u_0}{\partial y\partial z^2}+L_1\frac{\partial^3u_0}{\partial z\partial y^2}+L_1\frac{\partial^3u_0}{\partial z\partial x^2}\right)$$

$$\begin{aligned}&=H_1\frac{\partial f_0}{\partial x}+K_1\frac{\partial f_0}{\partial y}+L_1\frac{\partial f_0}{\partial z}-\ell^2\left(H_1\frac{\partial u_0}{\partial x}+K_1\frac{\partial u_0}{\partial y}+L_1\frac{\partial u_0}{\partial z}\right)\\&-H_1\Big[\frac{2}{\alpha_y\beta_yh_y^2}\left(\theta_{by}\frac{\partial u_2}{\partial x}-\beta_y\frac{\partial u_0}{\partial x}+\theta_{fy}\frac{\partial u_4}{\partial x}\right)+O(h_y^2)\Big]\\&-H_1\Big[\frac{2}{\alpha_z\beta_zh_z^2}\left(\theta_{bz}\frac{\partial u_5}{\partial x}-\beta_z\frac{\partial u_0}{\partial x}+\theta_{fz}\frac{\partial u_6}{\partial x}\right)+O(h_z^2)\Big]\\&-K_1\Big[\frac{2}{\alpha_x\beta_xh_x^2}\left(\theta_{bx}\frac{\partial u_1}{\partial y}-\beta_x\frac{\partial u_0}{\partial y}+\theta_{fx}\frac{\partial u_3}{\partial y}\right)+O(h_x^2)\Big]\\&-K_1\Big[\frac{2}{\alpha_z\beta_zh_z^2}\left(\theta_{bz}\frac{\partial u_5}{\partial y}-\beta_z\frac{\partial u_0}{\partial y}+\theta_{fz}\frac{\partial u_6}{\partial y}\right)+O(h_z^2)\Big]-\\&L_1\Big[\frac{2}{\alpha_x\beta_xh_x^2}\left(\theta_{bx}\frac{\partial u_1}{\partial z}-\beta_x\frac{\partial u_0}{\partial z}+\theta_{fx}\frac{\partial u_3}{\partial z}\right)+O(h_x^2)\Big]-\\&L_1\Big[\frac{2}{\alpha_y\beta_yh_y^2}\left(\theta_{by}\frac{\partial u_2}{\partial z}-\beta_y\frac{\partial u_0}{\partial z}+\theta_{fy}\frac{\partial u_4}{\partial z}\right)+O(h_y^2)\Big],\end{aligned}\tag{12}$$

and

$$\begin{aligned}&H_2\frac{\partial^4u_0}{\partial x^4}+K_2\frac{\partial^4u_0}{\partial y^4}+L_2\frac{\partial^4u_0}{\partial z^4}=\left(H_2\frac{\partial^2}{\partial x^2}+K_2\frac{\partial^2}{\partial y^2}+L_2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2u_0}{\partial x^2}+\frac{\partial^2u_0}{\partial y^2}+\frac{\partial^2u_0}{\partial z^2}\right)\\&-\Big[(H_2+K_2)\frac{\partial^4u_0}{\partial x^2\partial y^2}+(H_2+L_2)\frac{\partial^4u_0}{\partial x^2\partial z^2}+(K_2+L_2)\frac{\partial^4u_0}{\partial y^2\partial z^2}\Big],\\&H_2\frac{\partial^4u_0}{\partial x^4}+K_2\frac{\partial^4u_0}{\partial y^4}+L_2\frac{\partial^4u_0}{\partial z^4}=H_2\frac{\partial^2f_0}{\partial x^2}+K_2\frac{\partial^2f_0}{\partial y^2}+L_2\frac{\partial^2f_0}{\partial z^2}-\ell^2\left(H_2\frac{\partial^2u_0}{\partial x^2}+K_2\frac{\partial^2u_0}{\partial y^2}+L_2\frac{\partial^2u_0}{\partial z^2}\right)\\&-(H_2+K_2)\Big[\frac{2}{\alpha_y\beta_yh_y^2}\left(\theta_{by}\frac{\partial^2u_2}{\partial x^2}-\beta_y\frac{\partial^2u_0}{\partial x^2}+\theta_{fy}\frac{\partial^2u_4}{\partial x^2}\right)+O(h_y^2)\Big]\\&-(H_2+L_2)\Big[\frac{2}{\alpha_z\beta_zh_z^2}\left(\theta_{bz}\frac{\partial^2u_5}{\partial x^2}-\beta_z\frac{\partial^2u_0}{\partial x^2}+\theta_{fz}\frac{\partial^2u_6}{\partial x^2}\right)+O(h_z^2)\Big]\\&-(K_2+L_2)\Big[\frac{2}{\alpha_z\beta_zh_z^2}\left(\theta_{bz}\frac{\partial^2u_5}{\partial y^2}-\beta_z\frac{\partial^2u_0}{\partial y^2}+\theta_{fz}\frac{\partial^2u_6}{\partial y^2}\right)+O(h_z^2)\Big].\end{aligned}\tag{13}$$

Through central difference schemes the first order and second order derivative in equations (**Error! Reference source not found.**) and (13) can be approximated. Now combining equations (8) and (**Error! Reference source not found.**) with equations (**Error! Reference source not found.**) and (13), the nineteen point HOC scheme on nonuniform mesh points for the three dimensional Helmholtz equation (1) can be written as

$$\sum_{i=0}^{18}A_iu_i=f_0+H_1\frac{\partial f_0}{\partial x}+K_1\frac{\partial f_0}{\partial y}+L_1\frac{\partial f_0}{\partial z}+H_2\frac{\partial^2 f_0}{\partial x^2}+K_2\frac{\partial^2 f_0}{\partial x^2}+L_2\frac{\partial^2 f_0}{\partial z^2}.\tag{14}$$

The coefficients on the LHS in equation (14) are given as

$$A_0 = -2\Big(\frac{1}{\alpha_x h_x^2} + \frac{1}{\alpha_y h_y^2} + \frac{1}{\alpha_z h_z^2}\Big) - 2\Big(\frac{\ell^2 H_2}{\alpha_x h_x^2} + \frac{\ell^2 K_2}{\alpha_y h_y^2} + \frac{\ell^2 L_2}{\alpha_z h_z^2}\Big)$$

$$+4\Big(\frac{(H_2+K_2)}{\alpha_x\alpha_y h_x^2 h_y^2} + \frac{(K_2+L_2)}{\alpha_z\alpha_y h_z^2 h_y^2} + \frac{(H_2+L_2)}{\alpha_x\alpha_z h_x^2 h_z^2}\Big),$$

$$A_1 = 2\Big(\frac{\theta_{bx}}{\alpha_x\beta_x h_x^2} - \frac{H_1}{\alpha_y\beta_x h_x h_y^2} - \frac{H_1}{\alpha_z\beta_x h_x h_z^2} + \frac{\ell^2 H_2\theta_{bx}}{\alpha_x\beta_x h_x^2}\Big) + \frac{\ell^2 H_1}{\beta_x h_x} - 4\Big(\frac{(H_2+K_2)\theta_{bx}}{\alpha_x\alpha_y\beta_x h_x^2 h_y^2} + \frac{(H_2+L_2)\theta_{bx}}{\alpha_x\alpha_z\beta_x h_x^2 h_z^2}\Big),$$

$$A_2 = 2\Big(\frac{\theta_{by}}{\alpha_y\beta_y h_y^2} + \frac{\ell^2 K_2\theta_{by}}{\alpha_y\beta_y h_y^2} - \frac{K_1}{\alpha_x\beta_y h_x^2 h_y} - \frac{K_1}{\alpha_z\beta_y h_y h_z^2}\Big) + \frac{\ell^2 K_1}{\beta_y h_y} -$$

$$4\Big(\frac{(H_2+K_2)\theta_{by}}{\alpha_x\alpha_y\beta_y h_x^2 h_y^2} + \frac{(K_2+L_2)\theta_{by}}{\alpha_z\alpha_y\beta_y h_y^2 h_z^2}\Big),$$

$$A_3 = 2\Big(\frac{\theta_{fx}}{\alpha_x\beta_x h_x^2} + \frac{\ell^2 H_2\theta_{fx}}{\alpha_x\beta_x h_x^2} + \frac{H_1}{\alpha_y\beta_x h_x h_y^2} + \frac{H_1}{\alpha_z\beta_x h_x h_z^2}\Big) - \frac{\ell^2 H_1}{\beta_x h_x} - 4\Big(\frac{(H_2+K_2)\theta_{fx}}{\alpha_x\alpha_y\beta_x h_x^2 h_y^2} + \frac{(H_2+L_2)\theta_{fx}}{\alpha_x\alpha_z\beta_x h_x^2 h_z^2}\Big),$$

$$A_4 = 2\Big(\frac{\theta_{fy}}{\alpha_y\beta_y h_y^2} + \frac{\ell^2 K_2\theta_{fy}}{\alpha_y\beta_y h_y^2} + \frac{K_1}{\alpha_x\beta_y h_x^2 h_y} + \frac{K_1}{\alpha_z\beta_y h_y h_z^2}\Big) - \frac{\ell^2 K_1}{\beta_y h_y} - 4\Big(\frac{(H_2+K_2)\theta_{fy}}{\alpha_x\alpha_y\beta_y h_x^2 h_y^2} + \frac{(K_2+L_2)\theta_{fy}}{\alpha_y\alpha_z\beta_y h_y^2 h_z^2}\Big),$$

$$A_5 = 2\Big(\frac{\theta_{bz}}{\alpha_z\beta_z h_z^2} + \frac{\ell^2 L_2\theta_{bz}}{\alpha_z\beta_z h_z^2} - \frac{L_1}{\alpha_x\beta_z h_x^2 h_z} - \frac{L_1}{\alpha_z\beta_z h_y^2 h_z}\Big) + \frac{\ell^2 L_1}{\beta_z h_z} - 4\Big(\frac{(K_2+L_2)\theta_{bz}}{\alpha_y\alpha_z\beta_z h_y^2 h_z^2} + \frac{(H_2+L_2)\theta_{bz}}{\alpha_x\alpha_z\beta_z h_x^2 h_z^2}\Big),$$

$$A_6 = 2\Big(\frac{\theta_{fz}}{\alpha_z\beta_z h_z^2} + \frac{\ell^2 L_2\theta_{fz}}{\alpha_z\beta_z h_z^2} + \frac{L_1}{\alpha_x\beta_z h_x^2 h_z} + \frac{L_1}{\alpha_y\beta_z h_y^2 h_z}\Big) - \frac{\ell^2 L_1}{\beta_y h_y} -$$

$$4\Big(\frac{(K_2+L_2)\theta_{fz}}{\alpha_y\alpha_z\beta_z h_y^2 h_z^2} + \frac{(H_2+L_2)\theta_{fz}}{\alpha_x\alpha_z\beta_z h_x^2 h_z^2}\Big),$$

$$A_7 = \frac{2H_1\theta_{by}}{\alpha_y\beta_y\beta_x h_x h_y^2} + \frac{2K_1\theta_{bx}}{\alpha_x\beta_x\beta_y h_x^2 h_y} + \frac{4(H_2+K_2)\theta_{bx}\theta_{by}}{\alpha_x\alpha_y\beta_x\beta_y h_x^2 h_y^2},$$

$$A_8 = \frac{2K_1\theta_{fx}}{\alpha_x\beta_x\beta_y h_x^2 h_y} - \frac{2H_1\theta_{by}}{\alpha_y\beta_x\beta_y h_x h_y^2} + \frac{4(H_2+K_2)\theta_{by}\theta_{fx}}{\alpha_x\alpha_y\beta_x\beta_y h_x^2 h_y^2},$$

$$A_9 = \frac{-2K_1\theta_{fx}}{\alpha_x\beta_x\beta_y h_x^2 h_y} - \frac{2H_1\theta_{fy}}{\alpha_y\beta_x\beta_y h_x h_y^2} + \frac{4(H_2+K_2)\theta_{fx}\theta_{fy}}{\alpha_x\alpha_y\beta_x\beta_y h_x^2 h_y^2},$$

$$A_{10} = \frac{2H_1\theta_{fy}}{\alpha_y\beta_x\beta_y h_x h_y^2} - \frac{2K_1\theta_{bx}}{\alpha_x\beta_x\beta_y h_x^2 h_y} + \frac{4(H_2+K_2)\theta_{bx}\theta_{fy}}{\alpha_x\alpha_y\beta_x\beta_y h_x^2 h_y^2},$$

$$A_{11} = \frac{2H_1\theta_{bz}}{\alpha_z\beta_x\beta_z h_x h_z^2} + \frac{2L_1\theta_{bx}}{\alpha_x\beta_x\beta_z h_x^2 h_z} + \frac{4(H_2+L_2)\theta_{bx}\theta_{bz}}{\alpha_x\alpha_z\beta_x\beta_z h_x^2 h_z^2},$$

$$A_{12} = \frac{2K_1\theta_{bz}}{\alpha_z\beta_y\beta_z h_y h_z^2} + \frac{2L_1\theta_{by}}{\alpha_y\beta_y\beta_z h_y^2 h_z} + \frac{4(K_2+L_2)\theta_{by}\theta_{bz}}{\alpha_y\alpha_z\beta_y\beta_z h_y^2 h_z^2},$$

$$A_{13} = \frac{2L_1\theta_{fx}}{\alpha_x\beta_x\beta_z h_x^2 h_z} - \frac{2H_1\theta_{bz}}{\alpha_z\beta_x\beta_z h_x h_z^2} + \frac{4(H_2+L_2)\theta_{fx}\theta_{bz}}{\alpha_x\alpha_z\beta_x\beta_z h_x^2 h_z^2},$$

$$A_{14}=\frac{2L_1\theta_{fy}}{\alpha_y\beta_y\beta_z h_y^2 h_z}-\frac{2K_1\theta_{bz}}{\alpha_z\beta_y\beta_z h_y h_z^2}+\frac{4(K_2+L_2)\theta_{fy}\theta_{bz}}{\alpha_y\alpha_z\beta_y\beta_z h_y^2 h_z^2},$$

$$A_{15}=\frac{2H_1\theta_{fz}}{\alpha_z\beta_x\beta_z h_x h_z^2}-\frac{2L_1\theta_{bx}}{\alpha_x\beta_x\beta_z h_x^2 h_z}+\frac{4(H_2+L_2)\theta_{bx}\theta_{fz}}{\alpha_x\alpha_z\beta_x\beta_z h_x^2 h_z^2},$$

$$A_{16}=\frac{2K_1\theta_{fz}}{\alpha_z\beta_y\beta_z h_y h_z^2}-\frac{2L_1\theta_{by}}{\alpha_y\beta_y\beta_z h_y^2 h_z}+\frac{4(K_2+L_2)\theta_{by}\theta_{fz}}{\alpha_y\alpha_z\beta_y\beta_z h_y^2 h_z^2},$$

$$A_{17}=-\frac{2H_1\theta_{fz}}{\alpha_z\beta_x\beta_z h_x h_z^2}-\frac{2L_1\theta_{fx}}{\alpha_x\beta_x\beta_z h_x^2 h_z}+\frac{4(H_2+L_2)\theta_{fx}\theta_{fz}}{\alpha_x\alpha_z\beta_x\beta_z h_x^2 h_z^2},$$

$$A_{18}=-\frac{2K_1\theta_{fz}}{\alpha_z\beta_y\beta_z h_y h_z^2}-\frac{2L_1\theta_{fy}}{\alpha_y\beta_y\beta_z h_y^2 h_z}+\frac{4(K_2+L_2)\theta_{fy}\theta_{fz}}{\alpha_y\alpha_z\beta_y\beta_z h_y^2 h_z^2}.$$

It is easier to know that this scheme has fourth order of accuracy from expansion of t_0. Under uniform grids distribution the scheme has four to fifth order accuracy as proposed by [5, 8, 9].

3 Multigrid method

Multigrid method is one of the most efficient and fastest methods for solving PDEs. In multigrid method the rate of convergence is independent of the mesh size. This method is more effective for solving large scale of sparse linear systems obtained from the discretization of elliptic PDEs [1, 6, 12, 29, 31]. The main principle of multigrid method is to smooth the error on coarse grid level using basic iterative methods such as Jacobi or Gauss-Seidel method, etc. Multigrid method consists of three important components that are relaxation, restriction and interpolation operators. These are applied as " a single iteration of multigrid cycle comprised of manipulating the error by the application of relaxation method, fixing the residuals on the coarse grid level, solving the error equation on the coarse grid and adjusting the correction of coarse grid up to the fine grid level".

Some specific methods have been applied for the solution of 2D and 3D Helmholtz equation with HOC schemes on uniform grids [3, 9, 10, 14, 19, 22, 24]. A full weighting restriction operator and the standard bilinear interpolation operator are used as the inter grid transfer operators. But in case of nonuniform grids these restriction and interpolation operators are quite different.

3.1 Restriction operator

The principle of developing restriction operator is based on the evaluation of the residuals on the coarse grid level with the use of residuals on the fine grid level. In multigrid method Liu developed a law for the restriction of the residual [17], known as area law.

For every point on the coarse grid level, there are corresponding twenty six (26) fine grid points surrounding it. On coarse grid, there is a contribution of different degree between the reference grid points and the corresponding surrounding grid points on the fine grids; a full weighting restriction operator for nonuniform grids is constructed on the base of volume law. We have shown these points for convenience in Fig. 1(b). The basic idea for getting the full weighting restriction operator of each grid points is to analyze the weighting coefficients of the residuals. On the coarse grids the reference point (i, j, k) of the fine grids have the major contribution to it, so the corresponding weighting coefficient is evaluated by V_0/V. At that instant we noticed that the grid points near the reference point (I, j, k) have much more contributions than those far away from it. For instance, the weighting coefficient of the point $(i+1, j, k)$ is given by V_3/V and the point $(i-1, j, k)$ by V_1/V and so on. Now suppose that r_{ijk} is the residual at the fine grid point (i, j, k) and $\bar{r}_{\bar{i},\bar{j},\bar{k}}$ is the corresponding residual at the coarse grid point $(\bar{i}, \bar{j}, \bar{k})$. It is very simple to know that $i=2\bar{i}$, $j=2\bar{j}$ and $k=2\bar{k}$ thus the full weighting restriction operator on nonuniform grids can be written as

$$\begin{aligned}
\overline{r}_{\bar{i},\bar{j}} = \frac{1}{V}\Big(& V_0 r_{i,j,k} + V_1 r_{i-1,j,k} + V_2 r_{i,j-1,k} + V_3 r_{i+1,j,k} + V_4 r_{i,j+1,k} + V_5 r_{i,j,k-1} \\
& + V_6 r_{i,j,k+1} + V_7 r_{i-1,j-1,k} + V_8 r_{i+1,j-1,k} + V_9 r_{i+1,j+1,k} \\
& + V_{10} r_{i-1,j+1,k} + V_{11} r_{i-1,j,k-1} + V_{12} r_{i,j-1,k-1} + V_{13} r_{i+1,j,k-1} + V_{14} r_{i,j+1,k-1} \\
& + V_{15} r_{i-1,j,k+1} + V_{16} r_{i,j-1,k+1} + V_{17} r_{i+1,j,k+1} + V_{18} r_{i,j+1,k+1} \\
& + V_{19} r_{i-1,j-1,k-1} + V_{20} r_{i+1,j-1,k-1} + V_{21} r_{i+1,j+1,k-1} + V_{22} r_{i-1,j+1,k-1} \\
& + V_{23} r_{i-1,j-1,k+1} + V_{24} r_{i+1,j-1,k+1} \\
& + V_{25} r_{i+1,j+1,k+1} + V_{26} r_{i-1,j+1,k+1} \Big),
\end{aligned} \tag{15}$$

In which

$$V = (h_{fx}+h_{bx})\times(h_{fy}+h_{by})\times(h_{fz}+h_{bz}),\ V_0 = \frac{1}{8}(h_{fx}+h_{bx})\times(h_{fy}+h_{by})\times(h_{fz}+h_{bz}),$$

$$V_1 = \frac{1}{8}(h_{fx}\times(h_{fy}+h_{by})\times(h_{fz}+h_{bz})),\ V_2 = \frac{1}{8}(h_{fy}\times(h_{fx}+h_{bx})\times(h_{fz}+h_{bz})),$$

$$V_3 = \frac{1}{8}(h_{bx}\times(h_{fy}+h_{by})\times(h_{fz}+h_{bz})),\ V_4 = \frac{1}{8}(h_{by}\times(h_{fx}+h_{bx})\times(h_{fz}+h_{bz})),$$

$$V_5 = \frac{1}{8}(h_{fz}\times(h_{fx}+h_{bx})\times(h_{fy}+h_{by})),\ V_6 = \frac{1}{8}(h_{bz}\times(h_{fx}+h_{bx})\times(h_{fy}+h_{by})),$$

$$V_7 = \frac{1}{8}(h_{fx}\times h_{fy}\times(h_{fz}+h_{bz})),\ V_8 = \frac{1}{8}(h_{bx}\times h_{fy}\times(h_{fz}+h_{bz})),$$

$$V_9 = \frac{1}{8}(h_{bx}\times h_{by}\times(h_{fz}+h_{by})),\quad V_{10} = \frac{1}{8}(h_{fx}\times h_{by}\times(h_{fz}+h_{bz})),$$

$$V_{11} = \frac{1}{8}(h_{fx}\times h_{fz}\times(h_{fy}+h_{by})),\quad V_{12} = \frac{1}{8}(h_{fy}\times h_{fz}\times(h_{fx}+h_{bx})),$$

$$V_{13} = \frac{1}{8}(h_{bx}\times h_{fz}\times(h_{fy}+h_{by})),\ V_{14} = \frac{1}{8}(h_{by}\times h_{fz}\times(h_{fx}+h_{bx})),$$

$$V_{15} = \frac{1}{8}(h_{fx}\times h_{bz}\times(h_{fy}+h_{by})),\ V_{16} = \frac{1}{8}(h_{fy}\times h_{bz}\times(h_{fx}+h_{bx})),$$

$$V_{17} = \frac{1}{8}(h_{bx}\times h_{bz}\times(h_{fy}+h_{by})),\ V_{18} = \frac{1}{8}(h_{by}\times h_{bz}\times(h_{fx}+h_{bx})),$$

$$V_{19} = \frac{1}{8}(h_{fx}\times h_{fy}\times h_{fz}),\ V_{20} = \frac{1}{8}(h_{bx}\times h_{fy}\times h_{fz}),\ V_{21} = \frac{1}{8}(h_{bx}\times h_{by}\times h_{fz}),$$

$$V_{22} = \frac{1}{8}(h_{fx}\times h_{by}\times h_{fz}),\ V_{23} = \frac{1}{8}(h_{fx}\times h_{fy}\times h_{bz}),\ V_{24} = \frac{1}{8}(h_{bx}\times h_{fy}\times h_{bz}),$$

$$V_{25} = \frac{1}{8}(h_{bx}\times h_{by}\times h_{bz}),\ V_{26} = \frac{1}{8}(h_{fx}\times h_{by}\times h_{bz}),$$

If the step sizes reduce to equal size, then the whole volume V is divided into sixty-four equally small parts by the grid lines and half gird lines. Consider the Area of each part is ‘ $\overline{V}$ ’, we obtain that $V_0 = 8\overline{V}$, $V_1 = V_2 = \ldots = V_6 = 4\overline{V}$, $V_7 = V_8 = \ldots = V_{18} = 2\overline{V}$ and $V_{19} = V_{20} = \ldots = V_{26} = \overline{V}$. Due to this situation, the restriction operator will get reduced to the full weighting operator on equal mesh sizes [28].

$$\overline{r}_{\bar{i},\bar{j}} = \frac{1}{64}[8r_{i,j,k} + 4(r_{i-1,j,k} + r_{i+1,j,k} + r_{i,j+1,k} + r_{i,j-1,k} + r_{i,j,k+1} + r_{i,j,k-1}) + 2(r_{i+1,j+1,k} + r_{i-1,j+1,k} + r_{i+1,j-1,k} + r_{i-1,j-1,k}$$
$$+ r_{i,j+1,k-1} + r_{i,j+1,k+1} + r_{i,j-1,k-1} + r_{i,j-1,k+1} + r_{i+1,j,k-1} + r_{i+1,j,k+1} + r_{i-1,j,k-1} + r_{i-1,j,k+1}) + (r_{i+1,j+1,k-1} + r_{i-1,j+1,k+1}$$
$$+ r_{i+1,j-1,k-1} + r_{i-1,j-1,k-1} + r_{i+1,j+1,k+1} + r_{i-1,j+1,k-1} + r_{i+1,j-1,k+1} + r_{i-1,j-1,k+1})].$$

3.2 Interpolation operator

Similar strategy is used for constructing the interpolation operator. We observed that when grid points are shifted from coarse level to the fine level, at that instant the grids points on the coarse level are the grid points on fine level. These grid points are shifted directly from the coarse grid level to the fine grid level. Interpolation operator is expressed as $r_{i,j,k} = \overline{r}_{\bar{i},\bar{j},\bar{k}}$. Thus, the points on the fine grid are interpolated with their own neighboring points on the coarse level. The formula for error correction along *x*-, *y*- and *z* - directions are interpolated as

$$r_{i-1,j,k} = \frac{(h_{fx}\overline{r}_{\bar{i}-1,\bar{j},\bar{k}} + h_{bx}\overline{r}_{\bar{i},\bar{j},\bar{k}})}{h_{fx} + h_{bx}}, \quad r_{i,j-1,k} = \frac{(h_{fy}\overline{r}_{\bar{i},\bar{j}-1,\bar{k}} + h_{by}\overline{r}_{\bar{i},\bar{j},\bar{k}})}{h_{fy} + h_{by}}, \; r_{i,j,k-1} = \frac{(h_{fz}\overline{r}_{\bar{i},\bar{j},\bar{k}-1} + h_{bz}\overline{r}_{\bar{i},\bar{j},\bar{k}})}{h_{fz} + h_{bz}}.$$

In case of central grid points as in Fig.3(a), we use four grid points around them on the coarse grids to interpolate as following

$$r_{i-1,j-1,k} = \frac{1}{S_{xy}}\left(S_{1xy}\overline{r}_{\bar{i}-1,\bar{j}-1,k} + S_{2xy}\overline{r}_{\bar{i},\bar{j}-1,\bar{k}} + S_{3xy}\overline{r}_{\bar{i},\bar{j},\bar{k}} + S_{4xy}\overline{r}_{\bar{i}-1,\bar{j},\bar{k}}\right),$$

$$r_{i,j-1,k-1} = \frac{1}{S_{yz}}\left(S_{1yz}\overline{r}_{\bar{i},\bar{j}-1,\bar{k}-1} + S_{2yz}\overline{r}_{\bar{i},\bar{j},\bar{k}-1} + S_{3yz}\overline{r}_{\bar{i},\bar{j},\bar{k}} + S_{4yz}\overline{r}_{\bar{i},\bar{j}-1,\bar{k}}\right),$$

$$r_{i-1,j,k-1} = \frac{1}{S_{xz}}\left(S_{1xz}\overline{r}_{\bar{i}-1,\bar{j},\bar{k}-1} + S_{2xz}\overline{r}_{\bar{i},\bar{j},\bar{k}-1} + S_{3xz}\overline{r}_{\bar{i},\bar{j},\bar{k}} + S_{4xz}\overline{r}_{\bar{i}-1,\bar{j},\bar{k}}\right),$$

In which

$$S_{xy} = (h_{fx} + h_{bx}) \times (h_{fy} + h_{by}), S_{yz} = (h_{fy} + h_{by}) \times (h_{fz} + h_{bz}),$$

$$S_{xz} = (h_{fx} + h_{bx}) \times (h_{fz} + h_{bz}), S_{1xy} = h_{fx} \times h_{fy}, S_{2xy} = h_{bx} \times h_{fy}, S_{3xy} = h_{bx} \times h_{by},$$

$$S_{4xy} = h_{fx} \times h_{by}, \quad S_{1yz} = h_{fy} \times h_{fz}, \quad S_{2yz} = h_{by} \times h_{fz}, \quad S_{3yz} = h_{by} \times h_{bz}, \quad S_{4yz} = h_{fy} \times h_{bz},$$

$$S_{1xz} = h_{fx} \times h_{fz}, S_{2xz} = h_{bx} \times h_{fz}, S_{3xz} = h_{bx} \times h_{bz}, S_{4xz} = h_{fx} \times h_{bz}.$$

We use eight grids points around the central points in odd planes as in Fig. 3(b) on the coarse grids to make average as

$$r_{i-1,j-1,k-1} = \frac{1}{\tilde{V}}\left(\tilde{V}_1\overline{r}_{\bar{i}-1,\bar{j}-1,\bar{k}-1} + \tilde{V}_2\overline{r}_{\bar{i},\bar{j}-1,\bar{k}-1} + \tilde{V}_3\overline{r}_{\bar{i},\bar{j},\bar{k}-1} + \tilde{V}_4\overline{r}_{\bar{i}-1,\bar{j},\bar{k}-1} + \tilde{V}_5\overline{r}_{\bar{i}-1,\bar{j}-1,\bar{k}} + \tilde{V}_6\overline{r}_{\bar{i},\bar{j}-1,\bar{k}} + \tilde{V}_7\overline{r}_{\bar{i},\bar{j},\bar{k}} + \tilde{V}_8\overline{r}_{\bar{i}-1,\bar{j},\bar{k}}\right),$$

where $V_i^{\sim}$ for *i*=1,2,...,8, is the volume designed by all the grid lines around the reference points (*i*–1, *j*–1, *k*–1). $V^{\sim}$ is the total volume of $V_i^{\sim}$ for *i*=1,2,...,8, the formula for its calculation is

$$V^{\sim} = (h_{fx} + h_{bx}) \times (h_{fy} + h_{by}) \times (h_{fz} + h_{bz}), V_1^{\sim} = h_{fx} \times h_{fy} \times h_{fz},$$

$$V_2^{\sim} = h_{bx} \times h_{fy} \times h_{fz}, \; V_3^{\sim} = h_{bx} \times h_{by} \times h_{fz}, \; V_4^{\sim} = h_{fx} \times h_{by} \times h_{fz},$$

$$V_5^{\sim} = h_{fx} \times h_{fy} \times h_{bz}, \; V_6^{\sim} = h_{bx} \times h_{fy} \times h_{bz}, \; V_7^{\sim} = h_{bx} \times h_{by} \times h_{bz},$$

$$V_8{}^{\sim}=h_{fx}\times h_{by}\times h_{bz}.$$

When the grid spacing reduced to equal space, then the interpolation operator reduces to the trilinear interpolation on equal step sizes [28].

$$r_{i-1,j-1,k-1}=\frac{1}{\tilde{V}}\left(\tilde{V}_1\bar{r}_{\bar{i}-1,\bar{j}-1,\bar{k}-1}+\tilde{V}_2\bar{r}_{\bar{i},\bar{j}-1,\bar{k}-1}+\tilde{V}_3\bar{r}_{\bar{i},\bar{j},\bar{k}-1}+\tilde{V}_4\bar{r}_{\bar{i}-1,\bar{j},\bar{k}-1}+\tilde{V}_5\bar{r}_{\bar{i}-1,\bar{j}-1,\bar{k}}+\tilde{V}_6\bar{r}_{\bar{i},\bar{j}-1,\bar{k}}+\tilde{V}_7\bar{r}_{\bar{i},\bar{j},\bar{k}}+\tilde{V}_8\bar{r}_{\bar{i}-1,\bar{j},\bar{k}}\right),$$

$$r_{i,j,k-1}=\frac{1}{2}(\bar{r}_{\bar{i},\bar{j},\bar{k}-1}+\bar{r}_{\bar{i},\bar{j},\bar{k}}),\quad r_{i-1,j-1,k}=\frac{1}{4}(\bar{r}_{\bar{i}-1,\bar{j}-1,\bar{k}}+\bar{r}_{\bar{i},\bar{j}-1,\bar{k}}+\bar{r}_{\bar{i},\bar{j},\bar{k}}+\bar{r}_{\bar{i}-1,\bar{j},\bar{k}}),$$

$$r_{i,j-1,k-1}=\frac{1}{4}(\bar{r}_{\bar{i},\bar{j}-1,\bar{k}-1}+\bar{r}_{\bar{i},\bar{j},\bar{k}-1}+\bar{r}_{\overline{\bar{i},j,\bar{k}}}+\bar{r}_{\bar{i},\bar{j}-1,\bar{k}}),\quad r_{i-1,j,k-1}=\frac{1}{4}(\bar{r}_{\bar{i}-1,\bar{j},\bar{k}-1}+\bar{r}_{\bar{i},\bar{j},\bar{k}-1}+\bar{r}_{\bar{i},\bar{j},\bar{k}}+\bar{r}_{\bar{i}-1,\bar{j},\bar{k}}),$$

$$r_{i-1,j-1,k-1}=\frac{1}{8}(\bar{r}_{\bar{i}-1,\bar{j}-1,\bar{k}-1}+\bar{r}_{\bar{i},\bar{j}-1,\bar{k}-1}+\bar{r}_{\bar{i},\bar{j},\bar{k}-1}+\bar{r}_{\bar{i}-1,\bar{j},\bar{k}-1}+\bar{r}_{\bar{i}-1,\bar{j}-1,\bar{k}}+\bar{r}_{\bar{i},\bar{j}-1,\bar{k}}+\bar{r}_{\bar{i},\bar{j},\bar{k}}+\bar{r}_{\bar{i}-1,\bar{j},\bar{k}}),$$

3.3 Relaxation operator (Smoother)

In multigrid method, relaxation operator is an important operator. Its work is not to remove the errors, but to damp the high frequency components of the errors on the present grid level. Simple smoother (Gauss-Seidel relaxations) method can efficiently remove the errors in all directions for simple isotropic problems [18, 29], but in case of anisotropic and boundary layers problems, line Gauss-Seidel [6, 31] and alternating line Gauss-Seidel method [4, 7, 21, 33] are shown more robust smoothers. Plane relaxation is another efficient smoother which simultaneously updated all the grid points in a plane [20, 27].

4 Numerical experiments

To show the effectiveness of the considered schemes, we give some problems. V-cycle multigrid method is used with zero initial guess and the process is stopped when the Euclidean norm of the residual vector is reduced by 10^{-10} on the finest grid level. The effectiveness of multigrid method with HOC scheme and CD scheme (11) is presented. The reported errors are the maximum absolute errors between the computed solution and the exact solution on finest grid. The order of accuracy for a difference scheme is defined as,

$$Order=\log_2\frac{Error(N_1)}{Error(N_2)},$$

where $Error(N_1)$ and $Error(N_2)$ are maximum absolute errors approximated for two different grids with N_1+1 and N_2+1 points in both directions while N_1 is half of N_2.

Example 1 *Consider the following elliptic PDE with the source term*

$$u_{xx}+u_{yy}+u_{zz}+k^2u=f(x,y,z),\quad 0<x<1,\quad 0<y<1, \tag{16}$$

$$f(x,y,z)=(k^2+10^4e^{-100x})[y(1-y)x-z(1-z)]+200e^{-100x}, \tag{17}$$

the boundary conditions and the source function are given by the analytic solution, that is

$$u(x,y,z)=e^{-100x}[y(1-y)-x-z(z-1)].$$

The above problem has a steep boundary layer along x=0; therefore, we are using nonuniform grids along x- axis which are accumulating near x=0 and uniform grids along y,z -axis with the following stretching function [23],

$$x_i=\frac{i}{N_x}+\frac{1}{\text{p}}\sin(\frac{\text{p}i}{N_x}),\qquad y_j=\frac{j}{N_y},\qquad z_k=\frac{k}{N_z}$$

where λ is a stretching parameter and controlling the tightness of the grid points in x- direction. When λ<0, then more grid points are accumulated to the boundary x=0 and the boundary x=1 for λ>0. If λ=0 , then the grids reduce to be uniform. Also when the grid points are 32^2 and λ=0.8, then the distribution of grids in xy -plane is shown as in Fig. 1(a). The estimated accuracy and maximum absolute error with different stretching parameter λ is presented in Table 1. When λ=0, then the results are very poor. More accurate solution and order of convergence is obtained from HOC schemes with decreasing stretching parameter λ in nonuniform grids. We observed that when λ=0.8, the solution obtained with HOC schemes is more accurate, but when λ is decreases continuously to -0.9, then the accuracy does not further increase. This situation is not wondering because putting more grids in the boundary layer area that will necessarily cause lack of mesh points in the other regions in domain. indicates the configuration of solution in xy-plane with y=0.5, (a) shows the exact solution, (b) solution obtained from HOC scheme on uniform grids, (c) HOC scheme nonuniform grids, (d) computed solution with CDS scheme non uniform grids.

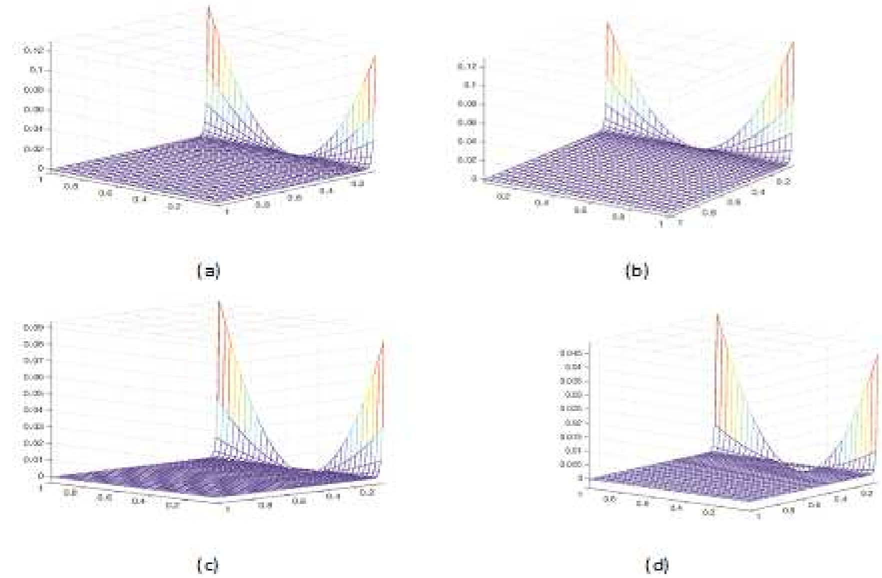


Figure 2: (a) Exact solution. (b) Computed solution from HOC scheme on uniform grids. (c) HOC scheme nonuniform grids. (d) CDS scheme with non uniform grids. The error vector $e_{ij} = u_{ij} - v_{ij}$ and N=32 are the number of nodes and k=10 and λ=0.8, for example 1.

Table 1. Comparison of maximum absolute errors, CPU timing and order of accuracy of the two schemes for example 1, $\| e_2 \|$, k=10, N=16,32,64,128 .

N	λ	16^2	32^2	64^2	128^2	CPU(seconds)	Order
CDS	0.0	$7.8052e^{-1}$	$6.9881e^{-1}$	$4.9012e^{-1}$	$5.4880e^{-2}$	0.590	1.88
	0.2	$5.4212e^{-1}$	$1.1012e^{-1}$	$7.1100e^{-2}$	$6.3876e^{-3}$	1.500	1.98
	0.4	$1.1184e^{-1}$	$7.1572e^{-2}$	$3.8067e^{-2}$	$1.0695e^{-3}$	1.610	1.99
	0.6	$7.8100e^{-2}$	$3.9180e^{-2}$	$5.8733e^{-3}$	$8.3434e^{-4}$	1.980	2.20
	0.8	$4.8327e^{-2}$	$6.1500e^{-3}$	$1.9150e^{-3}$	$1.8422e^{-4}$	2.430	2.60
	0.9	$8.4346e^{-3}$	$6.1984e^{-3}$	$6.8769e^{-4}$	$8.5500e^{-5}$	3.600	3.00
HOC	0.0	$4.8322e^{-1}$	$3.6630e^{-2}$	$4.4090e^{-2}$	$7.1218e^{-4}$	0.610	2.95
	0.2	$16.3118e^{-2}$	$9.8861e^{-3}$	$6.3039e^{-3}$	$8.8812e^{-5}$	0.700	3.10
	−0.4	$7.8711e^{-2}$	$2.6821e^{-3}$	$2.1025e^{-3}$	$1.0080e^{-5}$	1.840	3.84
	−0.6	$3.8237e^{-2}$	$6.8582e^{-3}$	$8.8101e^{-4}$	$6.5230e^{-6}$	2.800	3.95
	−0.8	$8.6227e^{-3}$	$3.8117e^{-4}$	$1.6056e^{-4}$	$5.4452e^{-7}$	3.220	4.00
	−0.9	$9.8844e^{-3}$	$6.5600e^{-4}$	$7.4371e^{-5}$	$9.4331e^{-7}$	4.350	4.25

Example 2 *Consider the PDE with a source term f(x, y),*

$$u_{xx} + u_{yy} + u_{zz} + k^2 u = f(x,y,z), \qquad 0 < x, y, z < 1, \tag{18}$$

with the analytic solution is

$$u(x,y,z) = \frac{(1-e^{100(x-1)})(1-e^{100(y-1)})(1-e^{100(z-1)})}{(1-e^{-100})^3}.$$

The source function is given by the analytic solution with the boundary layers along x=1, y=1 and z=1. Hence a nonuniform grids along both directions with accumulation near x=1, y=1 and z=1 is used by the following stretching formula

$$x_i = \frac{i}{N_x} + \frac{\lambda}{\pi} sin(\frac{\pi i}{N_x}),\ y_j = \frac{j}{N_y} + \frac{\lambda}{\pi} sin(\frac{\pi j}{N_y}),\ z_k = \frac{k}{N_z} + \frac{\lambda}{\pi} sin(\frac{\pi k}{N_z}).$$

When λ is going closer to 1, more grids are accumulated near x=1, y=1. When the grids size is 32^2 and λ=0.8, the distribution of grids is given as Fig. 5. Table 2 indicates the maximum absolute errors CPU timing and order of accuracy for different stretching parameter λ, for example 2. The value of λ changes from 0.0 to 0.9. We observed that in nonuniform grids with increasing the stretching parameter λ, more and more grids are accumulating into the boundary layers; consequently more accurate results are obtained from two different methods. Also the rate of convergence is continuously increases with the increase of λ. We observed that when λ=0.8, then considerably most accurate solution is obtained with the HOC scheme, but when λ is increased to 0.9, it leads to decrease in accuracy. Fig. 6 shows depicts contours of the exact solution in the plane for y=0.688. (a) Exact solution (b) by HOC on uniform grids, (c) computed solution by the CDS scheme on uniform grids, (d) solution by HOC on nonuniform grids with λ=0.8.

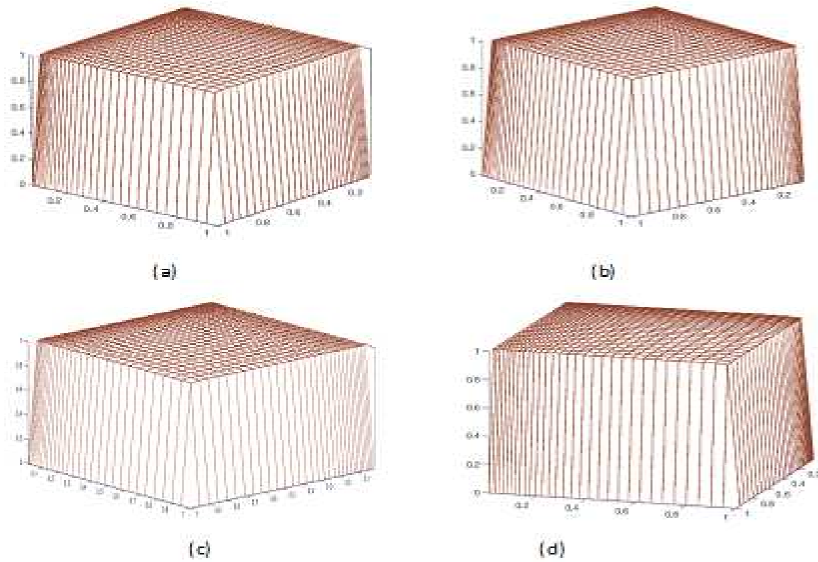


Figure 3: (a) Exact solution. (b) Computed solution by HOC scheme on uniform grids.
(c) CDS scheme on uniform grids (d) HOC scheme on nonuniform grids, 32^2, λ=0.8.

Table 2. Comparison of maximum absolute errors, CPU timing and order of accuracy of the two schemes for example 2, $\| e_2 \|$, *k=10, N=16,32,64,128.*

N	λ	16^2	32^2	64^2	128^2	CPU(second)	Order
CDS	0.0	$5.2002e^{-2}$	$4.8890e^{-2}$	$3.2290e^{-2}$	$7.1098e^{-3}$	0.109	1.44
	0.2	$3.4421e^{-2}$	$2.2213e^{-2}$	$1.1180e^{-2}$	$5.3112e^{-3}$	0.155	1.82
	0.4	$2.5541e^{-2}$	$1.7002e^{-2}$	$1.1040e^{-2}$	$3.1255e^{-4}$	0.161	1.98
	0.6	$1.1983e^{-2}$	$7.2100e^{-3}$	$4.1033e^{-3}$	$6.1043e^{-5}$	0.880	2.44
	0.8	$5.1287e^{-3}$	$1.9520e^{-3}$	$7.1530e^{-5}$	$1.6054e^{-5}$	2.221	2.88
	0.9	$2.4566e^{-4}$	$1.5424e^{-4}$	$7.8900e^{-6}$	$2.6041e^{-6}$	2.233	3.87
HOC	0.0	$4.2122e^{-2}$	$3.3211e^{-3}$	$7.4412e^{-4}$	$5.8234e^{-4}$	0.142	1.52
	0.2	$1.1218e^{-2}$	$1.6601e^{-3}$	$6.3318e^{-4}$	$4.7715e^{-5}$	0.160	1.86
	0.4	$9.7110e^{-3}$	$6.1218e^{-4}$	$3.6152e^{-5}$	$3.1961e^{-6}$	0.416	2.00
	0.6	$3.9328e^{-4}$	$4.1155e^{-5}$	$2.6001e^{-6}$	$2.3087e^{-6}$	1.808	2.48
	0.8	$3.7022e^{-5}$	$1.8676e^{-6}$	$5.2260e^{-7}$	$3.3350e^{-7}$	3.541	2.96
	0.9	$3.8844e^{-5}$	$2.8600e^{-6}$	$8.7210e^{-7}$	$5.3421e^{-7}$	4.322	3.90

5 Conclusion

A transformation free high-order compact finite difference scheme on nonuniform mesh sizes for discretizing a 3D Helmholtz equation. We have used special multigrid methods which solve the resulting system efficiently. It is observed that multigrid method with the Gauss-Seidel relaxation work very well in solving the high-order scheme discretized 3D Helmholtz equation. The HOC scheme has three to fourth order accuracy and is more efficient than CDS scheme. It is also observed that on uniform grids the HOC and CDS schemes can attain its maximum accuracy. But in case of boundary layer problems with suitable grid stretching ratios, the HOC scheme has the desired accuracy. Therefore it is reveal that grid distributions have much more significant effect on calculated accuracy for boundary layers problems. Numerical results shows that multigrid method with HOC has the required accuracy by the accumulating many more grid points into the boundary layer and faster than the CDS scheme. It is also obvious from the results that an increasing in the wave number *k*, overall error does not decrease.

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