

Characterizations of Quasimonotone of Multi mappings

Somayeh Yazdani

Department of Mathematics, Faculty of Basic Sciences, Nour Branch, Islamic Azad University, Nour, Iran. Received: March 1, 2015 Accepted: April 27, 2015

ABSTRACT

In this paper we introduce the concept of quasimonotone maps and some concepts of generalized monotonicity for multimappings, then establish connections between some concepts of generalized monotonicity for multimappings introduced during the last several years.

KEYWORDS: Quasiconvex functions, monoton maps, quasimonotone maps, generalized subdifferentials.

I. INTRODUCTION

The theory of monotone mappings in Banach spaces is of a recent origin. Some special results which now can be stated or interpreted in terms of this theory were obtained in the early 1950s for gradient mappings considered in the calculus of variations in Banach spaces. The explicit definition of the monotone mapping of a Banach space into its dual space which arose in a natural way from these investigations and was first introduced in a short note of Kacurovskii. This clearly showed that the theory of monotone mappings need not be restricted to gradient mappings and can be based on more primary structural properties of normed spaces. In recent years, operators which have some kind of generalized monotonicity property have a lot of attention. Many papers considering generalized monotonicity were devoted to the investigation of its relation to generalized convexity.

Monotonicity and convexity are closely connected. It is well known that if a function is convex then it^s (Clarke) generalized sub gradients are monotone; see Rockafellar [Rockafellar,1970] (a set-valued map $\Gamma: \mathbb{R}^n \to \mathbb{R}^n$ is monotone if whenever $y_i \in \Gamma(x_i)$, i = 1, 2, then $\langle y_1 - y_2, x_1 - x_2 \rangle \ge 0$, where $\langle y, z \rangle$ is the usual dot product in \mathbb{R}^n). The convers was established by F.H. Clarke in the particula case of a locally Lipschitzian function; see Clarke [Clarke,1983]. In [Poliquin,1990] Poliquin showed that a lower semi continuous function defined on \mathbb{R}^n is convex if (and only if) its Clarke sub differential is a monotone set-valued operator. Then for the particular case of a locally Lipschitzian function this fact had been remarked by Clarke [2, proposition 2.2.9].

Some characterizations of generalized convex functions are established by means of Clarke's subdifferential and directional derivatives. There is a characterization of the convexity of a function $f : X \rightarrow R$ via its

Clarke generalized sub differential $\partial^{\uparrow} f$: namely, X^* is convex if and only if $\partial^{\uparrow} f$ is monotone (see [Clarke,1983]). After the work in generalized monotonicity and the developments in the area of non smooth analyses, there has been an effort to characterize the generalized convexity of functions in terms of the generalized monotonicity of their sub differential. In particular, it was shown that a lower semi continuous function f is quasiconvex, if and only if its Clark -Rockafellar subdifferential is quasimonotone; under the further assumption that the function f is radially continuous, f is pseudoconvex if and only if its Clark-Rockafellar sub differential is pseudo monotone. In order to establish characterizations of generalized convexities of a function via natural properties of its generalized subdifferential (Minty, 1964), some concepts of generalized monotonicity for multimappings have been introduced in Ref [Schaible,1992]. This work establishes connections between monotone operators and convex function. Associated to each monotone operator, there is a family of convex functions, each of which characterizes the operator. The aim of this paper is to establish links for some other generalizations of convexity and monotonicity. The outline of the paper is as follows. In section 2, we establish some notation and recall the definitions and some results presented. In section 3, introduce some concepts of generalized monotonicity for multimappings and the equivalence between distinct concepts of quasimonotonicity proposed in the papers mentioned above is established.

^{*}Corresponding Author: Somayeh Yazdani, Department of Mathematics, Faculty of Basic Sciences, Nour Branch, Islamic Azad University, Nour, Iran. E-mail: S.yazdani200@yahoo.com

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II. BASIC DEFINITIONS and PRELIMINARY RESULTS

In the sequel, X is a Banach space with dueal space X^* . A multimapping F from X to X^* is said to be monotone (Ref. [Minty, 1964]) if

$$\langle x^*, y - x \rangle + \langle y^*, x - y \rangle \le 0, \quad \forall x, y \in X, \forall x^* \in F(x), \forall y^* \in F(y).$$
 (1)

We recall that a function f is said to be quasiconvex if its sublevel sets are convex, i.e.,

$$f(x + t(y-x)) \le \max \{f(x), f(y)\}, \qquad \forall x, y \in X, \forall t \in [0, 1].$$

Let X be a real topological vector space, X^* be its dual space, and $K \subseteq X$ be nonempty and convex. A multivalued map $T: K \tilde{A} X^*$ is called pseudomonotone (in the karamardian's sense) (Ref [Karamardian,1990]) if for every x, $y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$, the following implication holds:

$$\langle x^*, y - x \rangle \ge 0 \implies \langle y^*, y - x \rangle \ge 0$$

A monotone map is pseudomonotone, while a pseudomonotone map is quasimonotone. The converse is not true. The notation that we use is for the most part standard; however, let us provide a partial list for the reader s convenience. The spaces X and X^{*} are paired in duality by the continuous bilinear from $\langle x^*, x \rangle := x^*(x)$ Given $\varepsilon > 0, x \in X$ we denote by $B(x, \varepsilon)$ the closed ball centered at x with radius ε . Let $f : X \to \overline{R} := \{-\infty, +\infty\}$ be a given function. Assume that the value of the function is finite at a

point $x \in X$. The Clarke-Rockafellar generalized derivative of f at x in the direction v is defined by:

$$f^{\uparrow}(x,v) = \sup_{\varepsilon>0} \lim_{(y,\alpha)\downarrow_{f^x}} \sup_{t\downarrow 0} \inf_{u\in B(v,\varepsilon)} \frac{[f(y+tu)-\alpha]}{t},$$

Where $(y, \alpha) \downarrow_f x$ means that $y \to x, \alpha \to f(x), \alpha \ge f(y)$. When f is locally lipschitzian [Rockafellar, 1980], this derivative coincides with the Clarke directional derivative, which is defined by

$$f^{0}(x,v) = \lim_{z \to x, u \to v} \sup_{t \downarrow 0} \frac{[f(z+tu) - f(z)]}{t}$$

The circa-subdifferential (or Clarke-Rockafellar subdifferential) of f at x is

$$\partial^{\uparrow} f(x) = \{ x^* \in X^* : \langle x^*, v \rangle \leq f^{\uparrow}(x, v), \forall v \in X \},$$

with the convention that $\partial^{\uparrow} f(x)$ is empty if f is not finite at x. We need the following lemma, which was established in [Luc, 1983].

Lemma 2. 1. Assume that f is lower semicontinuos and that f(b) > f(a). Then, there exists a sequence (x_k) in X converging to some $x_0 \in [a,b)$, $x_k^* \in \partial^{\uparrow} f(x_k)$ such that, for any c = a + t(b-a) with $t \ge 1$ and for every k, one has $\langle x_k^*, c - x_k \rangle > 0$.

III. QUASIMONOTONICITY of MULTIMAPPING

The following definition is a generalized version of [Schaible, 1992 Definition 2.9 for multimappings] (or multifunctions or set-valued maps).

Definition 3.1. A multimapping $F : X \tilde{A} X^*$ is said to be quasimonotone if, for every pair of distinct points $x, y \in X$, one has

$$\exists x^* \in F(x) : \langle x^*, y - x \rangle > 0$$

$$\Rightarrow \forall y^* \in F(y) : \langle y^*, y - x \rangle \ge 0 \qquad (2)$$

In [Avriel, Diewert, et al., 1988, Proposition 2.8] Schaible has shown that a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if and only if ∇f is quasimonotone, and his result has been extended to nondifferentiable functions in [Hassouni].

Another notion of quasimonotonicity is the concept of directional quasimonotonicity of Hassouni given in [Ellaia, Definition 3.4]. Let us recall it and compare it with the preceding one.

Definition 3.2. A multimapping $F: X \tilde{A} X^*$ is said to be quasimonotone in the direction d if, for every $x \in X$ there exists $\lambda \in R := R \cup \{\pm \infty\}$ such that:

$$(\lambda - \overline{\lambda}) \langle v^*, d \rangle \ge 0,$$

$$\forall \lambda \in R, \forall v^* \in F(x + \lambda d).$$
(3)

F is called quasimonotone if it is quasimonotone in every direction d of X .

In [Ellaia, 1994 Theorem 3.10], it is shown that a Lipschitz function defined on a finite-dimensional space is quasiconvex if and only if $\partial^{\uparrow} f$ is quasimonotone (in any direction).

In comparison with the generalization from convexity to quasiconvexity, the following statement seems to be the most natural concept of quasimonotonicity of maps.

Definition 3.3. See [Luc, 1994 Definition 2.1]. A multimapping F is said to be quasimonotone if,

for every $x, y \in X, x^* \in F(x), y^* \in F(y)$ one has

$$\min\{\langle x^*, y - x \rangle, \langle y^*, x - y \rangle\} \le 0.$$
 (4)

In [Luc,1994 Definition 2.1], it is shown that a lower semicontinuous function is quasiconvex if and only if its generalized subdifferential is quasimonotone.

Theorem 3.1. The three preceding concepts are coincide.

Proof. Obviously (2) is equivalent to (4). Now, suppose that $F: X \tilde{A} X^*$ satisfies (3) for every $x, d \in X$. Let $x, y \in X$, and let $x^* \in F(x), y^* \in F(y)$ be given. We take d = y - x; from (3), if $\overline{\lambda} > 0$, by taking $\lambda = 0$, it follows that

$$\langle x^*, y - x \rangle \leq 0, \qquad \forall x^* \in F(x);$$

and if $\overline{\lambda} \leq 0$, by taking $\lambda = 1$, we get:

$$\langle y^*, x - y \rangle \le 0, \qquad \forall y^* \in F(y).$$

So, (4) follows.

Conversely, let (4) hold. For a given pair $(x, d) \in X^2$ and $t \in R$, let us set

$$H(t) = \{ \langle w^*, d \rangle : w^* \in F(x + td) \}, \qquad (5a)$$

$$T = \{t \in \mathbb{R} : H(t) \cap (0, +\infty) \neq \emptyset\}, \tag{5b}$$

$$T = \{t \in R : H(t) \cap (0, +\infty) \neq \emptyset\},$$
(5b)
$$S = \{s \in R : H(s) \cap (-\infty, 0) \neq \emptyset\}.$$
(5c)

We observe that, for any $t \in T, r > t$, one has $r \notin S$, since otherwise we could find $y^* \in F(y), z^* \in F(z)$, with y = x + td, z = x + rd, such that

$$\langle y^*, d \rangle > 0, \qquad \langle z^*, d \rangle > 0,$$

and the following would hold ;

$$\max\{\langle y^*, y-z \rangle, \langle z^*, z-y \rangle\} \\= \max\{(t-r)\langle y^*, d \rangle, (r-t)\langle z^*, d \rangle\} < 0,$$

a contradiction with (4).

Thus, for any pair $(t,s) \in T \times S$, we have $t \ge s$; hence,

Sup $S \leq \inf T$.

Take

$$\overline{\lambda} \ge \sup S, \quad \overline{\lambda} \le \inf T.$$

Let $\lambda \in R, v^* \in F(x + \lambda d)$. If $\lambda > \overline{\lambda}$, we have $\lambda \notin S$, hence

$$(\lambda - \lambda) \langle v^*, d \rangle \ge 0;$$

If $\lambda < \overline{\lambda}$, we have $\lambda \notin T$, hence

$$(\lambda - \overline{\lambda}) \langle v^*, d \rangle \geq 0,$$

so (3) is proved.

Corollary 3.1. If F is monoton, then F is quasimonotone in every direction [Hassouni].

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