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New Results in Q -Inner Product

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ABSTRACT

In this paper, we establish some new results in $(Q)-$ inner product. These are different from the results published in the book, "Semi-inner products and application" by S. S. Dragomir.

KEYWORDS: Inner product, Complex inner product, Q-inner product,

PRELIMINARIES

Dragomir (see, [1]-[4]) introduced some generalization of inner product in a real linear space that extends this concept in a different manner than the extension due to Lumer-Giles, Tapia or Miličić (see, [1]).

The following definitions are used by Dragomir in [1].

Definition: *A mapping* $(.,.,.,.)_q : X^4 \to \mathrm{R}$ *will be called a quaternary-inner product, or* $(Q)-$ *inner product, for short, if the following conditions are satisfied:*

(i) $(\alpha x_1 + \beta x_2, x_3, x_4, x_5)_q = \alpha(x_1, x_3, x_4, x_5)_q + \beta(x_2, x_3, x_4, x_5)_q$ *where* $\alpha, \beta \in \mathrm{R}$ *and* $x_i \in X\,(i = \overline{1,5})$.

(ii) $(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})_q = (x_1, x_2, x_3, x_4)_q$, *for any* σ *a permutation of the indices* $(1,2,3,4)$ *and* $x_i \in X\,(i = \overline{1,4})$,

(iii) *One has the following Schwartz type inequality*

$$\left|(x_1, x_2, x_3, x_4)_q\right|^4 \le \prod_{i=1}^{4}(x_i, x_i, x_i, x_i)_q, \tag{1.1}$$

for all $x_i \in X$, $(i = \overline{1,4})$ *and* $(x_1, x_1, x_1, x_1)_q > 0$ *if* $x_1 \ne 0$.

Definition *A real linear space* X *endowed with a* (Q) *-inner product* $(.,.,.,.)_q$ *on it will be called a* (Q) *-inner product space. Now by the definition of* (Q) *-inner product space, we can state the following simple properties:*

$$(0, x_2, x_3, x_4)_q = 0,$$

and

$$(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4)_q = \alpha^4 (x_1, x_2, x_3, x_4)_q,$$

for any $\alpha \in \mathrm{R}$ *and* $\forall\, x_1,\ x_2,\ x_3,\ x_4 \in X$.

Dragomir [1] also pointed out proposition that followed by the definition of $Q-$ inner product (see definition (01)) using two vectors.

Proposition: *Let* $(X, \|.\|_q)$ *be a* $Q-$ *normed space. Then for all* $x_1, x_2 \in X$ *, we have*

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$$\|x_1+x_2\|_q^4+\|x_1-x_2\|_q^4=2(\|x_1\|_q^4+\|x_2\|_q^4)+12(x_1,x_1,x_2,x_2)_q \tag{1.2}$$

and

$$\|x_1+x_2\|_q^4+\|x_1-x_2\|_q^4\le 2(\|x_1\|_q^4+\|x_2\|_q^4)+12\|x_1\|_q^2\|x_2\|_q^2. \tag{1.3}$$

MAIN RESULTS

Main object of this paper is to extend the idea of Dragomir [Drag1] given in equation (1.2) and inequality (1.3) from two vectors to four vectors

Proposition *Let* $(X,(.,.,.,.)_q)$ *be a* (Q) *-inner product space. Then the mapping*

$$\|.\|_q \;:\; X\to \mathrm{R},\ \|x\|_q=(x,x,x,x)_q^{\frac{1}{4}}$$

is a norm on X .

Proof: *Using definitions 1 and 2 of inner product and by simple calculation*

$$\begin{aligned}\|x_1+x_2+x_3+x_4\|_q^4 &= \sum_{i=1}^{4}\|x_i\|_q^4+4\sum_{i\neq j=1}^{4}\left(x_i,x_i,x_i,x_j\right)_q\\ &\quad +6\sum_{i\neq j=1}^{4}\left(x_i,x_i,x_j,x_j\right)_q+12\sum_{i\neq j\neq k=1}^{4}\left(x_i,x_i,x_j,x_k\right)_q\\ &\quad +\sum_{i\neq j\neq k\neq l=1}^{4}\left(x_i,x_j,x_k,x_l\right)_q,\end{aligned} \tag{2.1}$$

$\forall\ x_1,x_2,x_3,x_4\in X.$

From inequality (1.1)， *we have*

$$\left(x_i,x_i,x_i,x_j\right)_q\le\|x_i\|_q^3\|x_j\|_q, \tag{2.2}$$

$$\left(x_i,x_i,x_j,x_j\right)_q\le\|x_i\|_q^2\|x_j\|_q^2, \tag{2.3}$$

$$\left(x_i,x_i,x_j,x_k\right)_q\le\|x_i\|_q^2\|x_j\|_q\|x_k\|_q, \tag{2.4}$$

Using the eq.(2.2)-(2.4), equation (2.1) becomes

$$\begin{aligned}\|x_1+x_2+x_3+x_4\|_q^4 &\le \sum_{i=1}^{4}\|x_i\|_q^4+4\sum_{i\neq j=1}^{4}\left(\|x_i\|_q^3\|x_j\|_q\right)\\ &\quad +6\sum_{i\neq j=1}^{4}\left(\|x_i\|_q^2\|x_j\|_q^2\right)\\ &\quad +12\sum_{i\neq j\neq k=1}^{4}\left(\|x_i\|_q^2\|x_j\|_q\|x_k\|_q\right)\\ &\quad +\sum_{i\neq j\neq k\neq l=1}^{4}\left(\|x_i\|_q\|x_j\|_q\|x_k\|_q\|x_l\|_q\right)\end{aligned} \tag{2.5}$$

i.e

$$\|x_1+x_2+x_3+x_4\|_q^4 \le \left(\sum_{i=1}^{4}\|x_i\|_q\right)^4,$$

which produces the inequality

$$\|x_1+x_2+x_3+x_4\|_q \le \sum_{i=1}^{4}\|x_i\|_q.$$

on the other hand, we have

$$\|x_i\|_q \ge 0 \; \forall\, x_i, i=\overline{1,4}\in X,$$

and

$$\|x_i\|_q = 0 \;\Rightarrow\; x_i = 0,$$

and finally, we also have:

$$\|\alpha x_i\|_q = |\alpha|\|x_i\|_q, \text{ where } \alpha \in \mathrm{R} \text{ and } x_i \in X.$$

Consequently $\|.\|_q$ *is a norm and the proposition is proved.*

Proposition *Let* $(X,(.,.,.,.)_q)$ *be a* $Q-$ *normed space. Then for all* $x_1,x_2,x_3,x_4 \in X$ *, we have:*

$$\begin{aligned}
&\|x_1+x_2+x_3+x_4\|_q^4+\|x_1+x_2-x_3-x_4\|_q^4+\|x_1-x_2+x_3-x_4\|_q^4\\
&+\|x_1-x_2-x_3+x_4\|_q^4+\|x_1+x_2+x_3-x_4\|_q^4+\|x_1+x_2-x_3+x_4\|_q^4\\
&+\|x_1-x_2+x_3+x_4\|_q^4+\|x_4+x_2+x_3-x_1\|_q^4 \qquad (2.6)\\
&=8\sum_{i=1}^{4}\|x_i\|_q^4+48\sum_{i\neq j=1}^{4}\left(x_i,x_i,x_j,x_j\right)_q.
\end{aligned}$$

Proof : We know from (2.1) that

$$\begin{aligned}
\|x_1+x_2+x_3+x_4\|_q^4 = &\sum_{i=1}^{4}\|x_i\|_q^4+4\sum_{i\neq j=1}^{4}\left(x_i,x_i,x_i,x_j\right)_q\\
&+6\sum_{i\neq j=1}^{4}\left(x_i,x_i,x_j,x_j\right)_q+12\sum_{i\neq j\neq k=1}^{4}\left(x_i,x_i,x_j,x_k\right)_q\\
&+\sum_{i\neq j\neq k\neq l=1}^{4}\left(x_i,x_j,x_k,x_l\right)_q
\end{aligned}$$

Similarly computing other seven terms in (2.6) and adding we obtain the required result.

Proposition: *Let* $(X,(.,.,.,.)_q)$ *be a* $Q-$ *normed space. Then for all* $x_1,x_2,x_3,x_4 \in X,$ *we have:*

$$
\begin{aligned}
&\|x_1+x_2+x_3+x_4\|_q^4+\|x_1+x_2-x_3-x_4\|_q^4+\|x_1-x_2+x_3-x_4\|_q^4 \\
&+\|x_1-x_2-x_3+x_4\|_q^4+\|x_1+x_2+x_3-x_4\|_q^4+\|x_1+x_2-x_3+x_4\|_q^4 \\
&+\|x_1-x_2+x_3+x_4\|_q^4+\|x_4+x_2+x_3-x_1\|_q^4 \\
&\leq 8\sum_{i=1}^{4}\|x_i\|_q^4+48\sum_{i\neq j=1}^{4}\left(\|x_i\|_q^2\|x_j\|_q^2\right)
\end{aligned}
\tag{2.7}
$$

Proof : From proposition 05 and using following inequalities

$$(x_1,x_1,x_2,x_2)_q \leq \|x_1\|_q^2\|x_2\|_q^2,$$

$$(x_1,x_1,x_3,x_3)_q \leq \|x_1\|_q^2\|x_3\|_q^2,$$

$$(x_1,x_1,x_4,x_4)_q \leq \|x_1\|_q^2\|x_4\|_q^2,$$

$$(x_2,x_2,x_3,x_3)_q \leq \|x_2\|_q^2\|x_3\|_q^2,$$

$$(x_2,x_2,x_4,x_4)_q \leq \|x_2\|_q^2\|x_4\|_q^2,$$

$$(x_3,x_3,x_4,x_4)_q \leq \|x_3\|_q^2\|x_4\|_q^2,$$

we obtain the required (2.7).

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