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New Results in SQ -Inner Product

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ABSTRACT

In this paper, we establish some new results in $(SQ)-$ inner product. These are different from the results published in the book of S. S. Dragomir titled "Semi- Inner Products and Application".

KEYWORDS: Inner product, Complex inner product, SQ-inner product,

PRELIMINARIES

Dragomir (see, [2] and [3]) introduced some generalization of inner product in a real or complex linear space. The following definitions are used by Dragomir in [2].

Definition *A mapping* $(.,.,.,.)_{sq} : X^4 \longrightarrow \mathbb{k}(\mathbb{k} = \mathbb{R}, \mathbb{C})$ *is said to be a sesqui- quaternary-inner product, or* $(SQ)-$ *inner product, for short, if the following conditions are satisfied:*

(i) $(\alpha x_1 + \beta x_2, x_3, x_4, x_5)_{sq} = \alpha(x_1, x_3, x_4, x_5)_{sq} + \beta(x_2, x_3, x_4, x_5)_{sq}$ *where* $\alpha, \beta \in \mathbb{k}$ *and* $x_i \in X\,(i = \overline{1,5})$,

$(ii)\overline{(x_1, x_2, x_3, x_4)_{sq}} = (x_2, x_1, x_4, x_3)_{sq}$,

$(iii)(x_1, x_2, x_3, x_4)_{sq} = (x_3, x_4, x_1, x_2)_{sq}$,

$(iv)(x_1, x_1, x_1, x_1)_{sq} > 0$ *if* $x_1 \in X, x_1 \neq 0$,

(v) *One has the following Schwartz type inequality*

$$\left|(x_1, x_2, x_3, x_4)_{sq}\right|^4 \leq \prod_{i=1}^{4} (x_i, x_i, x_i, x_i)_{sq}, \tag{1.1}$$

for all $x_i \in X$, $(i = \overline{1,4})$.

By the definition of $(SQ)-$ inner product, it is easy to see that $(.,.,.,.)_{sq}$ is linear in the third variable and antilinear in the second and fourth variables and the number $(x_1, x_1, x_2, x_2)_{sq}$ is real for every $x_1, x_2 \in X$.

Note that $(Q)-$ inner product space is a $(SQ)-$ inner product.

Dragomir [2] also pointed out the following propositions that followed by the definition of $SQ-$ inner product (see definition 01 given above) using two vectors.

Proposition *Let* $(X, \|.\|_q)$ *be a* $SQ-$ *normed space. Then for all* $x_1, x_2 \in X$ *, we have*

$$\|x_1 + x_2\|_{sq}^4 + \|x_1 - x_2\|_{sq}^4 = 2(\|x_1\|_{sq}^4 + \|x_2\|_{sq}^4) + 4(x_1, x_1, x_2, x_2)_{sq}$$
$$+4\,\mathrm{Re}(x_1, x_2, x_1, x_2)_{sq} + 4\,\mathrm{Re}(x_1, x_2, x_2, x_1)_{sq}$$

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and

$$\|x_1+x_2\|_{sq}^4+\|x_1-x_2\|_{sq}^4 \le 2(\|x_1\|_{sq}^4+\|x_2\|_{sq}^4)+12\|x_1\|_{sq}^2\|x_2\|_{sq}^2.$$

MAIN RESULTS

Using the theories given in ([1] - [6]), here our main objective is to extend the idea of Dragomir [2] given in equation (1.2) and inequality (1.3) from two vectors to four vectors.

Proposition *Let* $(X,(.,.,.,.)_{sq})$ *be a* (SQ) *-inner product space. Then the mapping*

$$\|.\|_{sq}\ :\ X \to \mathrm{R},\ \|x\|_{sq}=(x,x,x,x)_{sq}^{\frac{1}{4}},$$

is a norm on X .

Proof *Using definition **1** of inner product, we can easily calculate*

$$\begin{aligned}\|x_1+x_2+x_3+x_4\|_{sq}^4 &= \sum_{i=1}^{4}\|x_i\|_{sq}^4 + 4\sum_{i\neq j=1}^{4}\mathrm{Re}(x_i,x_i,x_i,x_j)_{sq}\\ &+2\sum_{i\neq j=1}^{4}(x_i,x_i,x_j,x_j)_{sq}+2\sum_{i\neq j=1}^{4}\mathrm{Re}(x_j,x_i,x_j,x_i)_{sq}\\ &+2\sum_{i\neq j=1}^{4}\mathrm{Re}(x_i,x_j,x_j,x_i)_{sq}+2\sum_{i\neq j\neq k=1}^{4}\mathrm{Re}(x_i,x_i,x_j,x_k)_{sq}\\ &+2\sum_{i\neq j\neq k=1}^{4}\mathrm{Re}(x_i,x_j,x_k,x_i)_{sq}+\sum_{i\neq j\neq k=1}^{4}\mathrm{Re}(x_j,x_i,x_k,x_i)_{sq}\\ &+4\,\mathrm{Re}(x_1,x_2,x_3,x_4)_{sq}+4\,\mathrm{Re}(x_1,x_2,x_4,x_3)_{sq}\\ &+4\,\mathrm{Re}(x_1,x_3,x_2,x_4)_{sq}+4\,\mathrm{Re}(x_4,x_2,x_1,x_3)_{sq}\\ &+4\,\mathrm{Re}(x_2,x_3,x_1,x_4)_{sq}+4\,\mathrm{Re}(x_3,x_2,x_1,x_4)_{sq}\end{aligned} \tag{2.1}$$

$\forall\ x_1,x_2,x_3,x_4 \in X.$

Using the inequality (1.1), we have

$$\mathrm{Re}(x_i,x_i,x_i,x_j)_{sq} \le \|x_i\|_{sq}^3\|x_j\|_{sq}, \tag{2.2}$$

$$\mathrm{Re}(x_i,x_i,x_j,x_j)_{sq} \le \|x_i\|_{sq}^2\|x_j\|_{sq}^2, \tag{2.3}$$

$$\mathrm{Re}(x_i,x_i,x_j,x_k)_{sq} \le \|x_i\|_{sq}^2\|x_j\|_{sq}\|x_k\|_{sq}. \tag{2.4}$$

Equation.(1.1) becomes

$$\begin{aligned}\|x_1+x_2+x_3+x_4\|_{sq}^4 \le \sum_{i=1}^{4}\|x_i\|_{sq} &+ 4\sum_{i\neq j=1}^{4}\left(\|x_i\|_{sq}^3\|x_j\|_{sq}\right)\\ &+6\sum_{i\neq j=1}^{4}\left(\|x_i\|_{sq}^2\|x_j\|_{sq}^2\right)\\ &+12\sum_{i\neq j\neq k=1}^{4}\left(\|x_i\|_{sq}^2\|x_j\|_{sq}\|x_k\|_{sq}\right)\\ &+\sum_{i\neq j\neq k\neq l=1}^{4}\left(\|x_i\|_{sq}\|x_j\|_{sq}\|x_k\|_{sq}\|x_l\|_{sq}\right)\end{aligned} \tag{2.5}$$

implies

$$\|x_1+x_2+x_3+x_4\|_{sq}^4 \le \left(\sum_{i=1}^{4}\|x_i\|_{sq}\right)^4$$

which produces the inequality

$$\|x_1+x_2+x_3+x_4\|_{sq} \le \sum_{i=1}^{4}\|x_i\|_{sq},$$

on the other hand, we have

$$\|x_i\|_{sq} \ge 0 \ \forall\, x_i, i=\overline{1,4} \in X$$

and

$$\|x_i\|_{sq} = 0 \Rightarrow x_i = 0$$

and finally, we also have :

$$\|\alpha x_i\|_{sq} = |\alpha|\,\|x_i\|_{sq}, \text{ where } \alpha \in \mathrm{R} \text{ and } x_i \in X$$

Consequently $\|.\|_{sq}$ *is a norm and the proposition 03 is proved.*

Proposition: *Let* $(X,(.,.,.,.)_{sq})$ *be a* SQ *normed space. Then for all* $x_1,x_2,x_3,x_4 \in X$ *, we have:*

$$\begin{aligned}
&\|x_1+x_2+x_3+x_4\|_{sq}^4+\|x_1+x_2-x_3-x_4\|_{sq}^4\\
&\quad+\|x_1-x_2+x_3-x_4\|_{sq}^4+\|x_1-x_2-x_3+x_4\|_{sq}^4\\
&\quad+\|x_1+x_2+x_3-x_4\|_{sq}^4+\|x_1+x_2-x_3+x_4\|_{sq}^4\\
&\quad+\|x_1-x_2+x_3+x_4\|_{sq}^4+\|x_4+x_2+x_3-x_1\|_{sq}^4 \qquad (2.6)\\
&=8\sum_{i=1}^{4}\|x_i\|_{sq}^4+16\sum_{i\ne j=1}^{4}(x_i,x_i,x_j,x_j)_{sq}\\
&\quad+16\sum_{i\ne j=1}^{4}\mathrm{Re}(x_j,x_i,x_j,x_i)_{sq}+16\sum_{i\ne j=1}^{4}\mathrm{Re}(x_i,x_j,x_j,x_i)_{sq}
\end{aligned}$$

Proof : We know that

$$\begin{aligned}
&\|x_1+x_2+x_3+x_4\|_{sq}^4\\
&=\sum_{i=1}^{4}\|x_i\|_{sq}^4+4\sum_{i\ne j=1}^{4}\mathrm{Re}(x_i,x_i,x_i,x_j)_{sq}\\
&\quad+2\sum_{i\ne j=1}^{4}(x_i,x_i,x_j,x_j)_{sq}+2\sum_{i\ne j=1}^{4}\mathrm{Re}(x_j,x_i,x_j,x_i)_{sq}\\
&\quad+2\sum_{i\ne j=1}^{4}\mathrm{Re}(x_i,x_j,x_j,x_i)_{sq}+2\sum_{i\ne j\ne k=1}^{4}\mathrm{Re}(x_i,x_i,x_j,x_k)_{sq} \qquad (2.7)\\
&\quad+2\sum_{i\ne j\ne k=1}^{4}\mathrm{Re}(x_i,x_j,x_k,x_i)_{sq}+\sum_{i\ne j\ne k=1}^{4}\mathrm{Re}(x_j,x_i,x_k,x_i)_{sq}\\
&\quad+4\,\mathrm{Re}(x_1,x_2,x_3,x_4)_{sq}+4\,\mathrm{Re}(x_1,x_2,x_4,x_3)_{sq}\\
&\quad+4\,\mathrm{Re}(x_1,x_3,x_2,x_4)_{sq}+4\,\mathrm{Re}(x_4,x_2,x_1,x_3)_{sq}\\
&\quad+4\,\mathrm{Re}(x_2,x_3,x_1,x_4)_{sq}+4\,\mathrm{Re}(x_3,x_2,x_1,x_4)_{sq}.
\end{aligned}$$

Computing other terms of the type $\left\| x_1 \pm x_2 \pm x_3 \pm x_4 \right\|_{sq}^4$ adding and simplifying the results, we obtain the required result (2.6).

Proposition: *Let* $(X,(.,.,.,.)_{sq})$ *be a* $SQ-$ *normed space. Then for all* $x_1, x_2, x_3, x_4 \in X$ *we have:*

$$\begin{aligned}
&\left\| x_1 + x_2 + x_3 + x_4 \right\|_{sq}^4 + \left\| x_1 + x_2 - x_3 - x_4 \right\|_{sq}^4 \\
&\quad + \left\| x_1 - x_2 + x_3 - x_4 \right\|_{sq} + \left\| x_1 - x_2 - x_3 + x_4 \right\|_{sq}^4 \\
&\quad + \left\| x_1 + x_2 + x_3 - x_4 \right\|_{sq}^4 + \left\| x_1 + x_2 - x_3 + x_4 \right\|_{sq}^4 \\
&\quad + \left\| x_1 - x_2 + x_3 + x_4 \right\|_{sq}^4 + \left\| x_4 + x_2 + x_3 - x_1 \right\|_{sq}^4 \\
&\leq 8\sum_{i=1}^{4} \left\| x_i \right\|_{sq}^4 + 48 \sum_{i \neq j = 1}^{4} \left\| x_i \right\|_{sq}^2 \left\| x_j \right\|_{sq}^2 .
\end{aligned} \tag{2.8}$$

Proof : From proposition 4 and using the inequalities

$$\operatorname{Re}\left(x_i, x_i, x_j, x_j\right)_{sq} \leq \left\| x_i \right\|_{sq}^2 \left\| x_j \right\|_{sq}^2,$$

$$\operatorname{Re}\left(x_j, x_i, x_j, x_i\right)_{sq} \leq \left\| x_j \right\|_{sq}^2 \left\| x_i \right\|_{sq}^2,$$

$$\operatorname{Re}\left(x_i, x_j, x_j, x_i\right)_{sq} \leq \left\| x_i \right\|_{sq}^2 \left\| x_j \right\|_{sq}^2,$$

we obtain the required inequality (2.8).

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