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Application of Homotopy Perturbation Method to an SIR Epidemic Model

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ABSTRACT

In this paper, an SIR (Susceptible, Infected and Recovered) epidemic model is consider to find the approximate/ exact solution by Perturbation method. We apply the Homotopy perturbation method to the model to find its approximate solution. The Homotopperturbation method gives a good results for non-linear problems of differential equations. First, we explain the method in detail and then we apply the HPM to our model. The analytical results are solve numerically and compare with other standard methods. The numerical results shows the that HPM have a good agreement with other standards. For illustrations of the theoretical results numerical results are presented in the form of Graphs.

KEYWORDS: SIR model, Homotopy Perturbation Method, Numerical Simulations

INTRODUCTION

Mathematical modeling has an important tools in analyzing the spread and control of infectious diseases. Mathematical models take into account main factors that govern development of a disease, such as transmission and recovery rates, and predict how the disease will spread over a period of time. In recent years, many attempts have been made to develop realistic mathematical models for investigating the transmission dynamics of infectious diseases, and the asymptotic behaviors of these epidemic models are studied [1]. One of the main issues in the study of behavior of epidemic model is the analysis of steady states and their stability [2]. When the population in each compartment does not exhibit any structure (as space location, age, etc.) and no delayed processes are considered, the time evolution of such three compartments is described by three ordinary differential equations, namely an SIR (susceptible, infectious, recovered) models [3, 4, 5, 15-20].

In the research literature, it was frequently assumed that the disease incubation is negligible. In this case, once infected, each susceptible individual becomes infectious instantaneously and later recovers with a temporary acquired immunity. Most of the biological problems are inherently nonlinear. The Scientist is in search to find such Numerical methods or Perturbation method to find the exact solution/approximate solution to these non-linear problems. Most of the non-linear problems have not solved exactly to find their exact solution, only a few number of non-linear of problems have the exact solution. So these non-linear problems can be solved by Numerical or traditional methods. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results. In the analytical perturbation method, we need to exert the small parameter in the equation. Therefore, finding the small parameter and exerting it into the equation are difficulties of this method. Since there are some limitations with the common perturbation method, and also because the basis of the common perturbation method is upon the existence of a small parameter, developing the method for different applications is very difficult. Therefore, many different powerful mathematical methods have been recently introduced to vanish the small parameters, such as artificial parameter method [6, 7]. Zaman considered the model [8] to study the approximate solution with HPM and compare with other standards methods.

The Homotopy Analysis Method is one of the well-known methods to solve the nonlinear equations. In the last decade, the idea of Homotopy was combined with perturbation. The fundamental work was done by Liao and He's. This method involves a free parameter, whose suitable choice results into fast convergence. He introduced Homotopy Perturbation Method and its application in several problems see for example and the references there in [9, 10, 11] while Ali et al. [12] presented the solution of multi points boundary values byusing Optimal Homotopy Analysis Method. These methods are independent of the assumption of small parameter as well as they cover all the

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advantages of the perturbation method.
In this paper, we consider the model presented in [13] by applying the Homotopy perturbation method, to find the approximate solution. First, we formulate our problem and then apply the HPM to find the analytical as well as numerical solutions.
The paper is organized as follows: Section 2 is devoted to the mathematical formulation of the model and basic idea of HPM. In Section 3 the model is solve by HPM. In Section 4 we present the solution of the numerical solution of model. At last the conclusion and references are presented.

2. BASIC IDEA OF HOMOTOPY PERTURBATION METHOD AND MODEL FRAMEWORK

In this section, first we explain the Homotopy perturbation method in detail and then we apply the technique of HPM to our proposed epidemic model. Homotopy perturbation method was first time introduced by the He [6, 14] for solving the nonlinear differential equations problems.

$$B(m) = f(d), \qquad d \in \Lambda \tag{1}$$

with the boundary conditions

$$\psi\left(m, \frac{\partial m}{\partial n}\right) = 0, \qquad d \in \Omega \tag{2}$$

Here B represent the general differential operator, ψ is the boundary operator, the analytic function is f(d) , Ω is the boundary of the domain Λ and $\frac{\partial}{\partial n}$ represent the differentiation along the normal vector drawn outward Λ . The operator B is divided in two parts, i.e, H is linear and K is nonlinear. So we write equation namely by (3) in the following form:

$$H(m)+K(m)=f(d). \tag{3}$$

Define the, Homotopy $v(r, p) : \Lambda \times [0,1]$™ $\Re$, that satisfies,

$$F(v, p) = (1- p)\left[H(v) - H(m_o)\right] + p\left[B(v) - f(d)\right] = 0 \tag{4}$$

Also we write in simplified form:

$$F(v,p)=H(v)+pH(m\_o)+p[K(m)-f(d)]=0 \tag{5}$$

Here m_o shows the initial approximation of (5) and p is the embedding parameter, $p \in [0,1]$. It is obvious that

$$F(v,0) = \left[H(v) - H(m_o)\right] = 0, \; F(v,1) = \left[B(v) - f(d)\right] = 0 \tag{6}$$

For p=0, we get $\left[H(v) - H(m_o)\right]$ and p=1 we get, $F(v,1) = \left[B(v) - f(d)\right]$ Applying the perturbation technique consider p is the small parameter then the solution of equation 4 can be considered as series in p, which is given by

$$v = v_o + pv_1 + p^2 v_2 + p^3 v_3 + ..., \tag{7}$$

when p approaches 1 the equation (4) becomes the original equation (3) and (7) becomes the approximate solution of (3) given by

$$m = \lim_{p™ 1} v = v_o + pv_1 + p^2 v_2 + p^3 v_3 + ..., \tag{8}$$

The series (8) is convergent for most of cases see [6, 14].

Now we formulate our problem here, we assume that S(t) represents number of susceptible human at time t; I(t) represents number of infected human in the population at time t; R(t) represents number of recovered human in the population at time t. We denote the total population size by N(t), with N(t)=S(t)+I(t)+R(t). To understand the dynamic of the disease we formulate the model, which consists of a non-linear system of differential equations in the following:

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(1-\frac{S}{k}) - \frac{\beta S(t)I(t)}{1+\alpha S}, \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1+\alpha S} - (\mu_1 + \gamma)I(t), \\ \frac{dI(t)}{dt} &= \gamma I(t) - \mu_2 R(t), \end{aligned} \tag{9}$$

with initial conditions

$S(0) \geq 0, \quad I(0) \geq 0, \quad R(0) \geq 0$ (10)

The parameters and their definitions are represented in Table.1.

3. SOLUTION OF THE MODEL BY HPM

In this section, we apply the Homotopy perturbation method to our proposed model (9) in the following.

$$\begin{aligned} &\mathrm{L}S(t) - \mathrm{L}S^{O}(t) = q(rS(1-\frac{S}{k}) - \frac{\beta S(t)I(t)}{1+\alpha S} - \mathrm{L}S^{O}(t)), \\ &\mathrm{L}I(t) - \mathrm{L}I^{O}(t) = q(\frac{\beta S(t)I(t)}{1+\alpha S} - (\mu_1+\gamma)I(t) - \mathrm{L}I^{O}(t)), \\ &\mathrm{L}R(t) - \mathrm{L}R^{O}(t) = q(\gamma I(t) - \mu_2 R(t) - \mathrm{L}R^{O}(t)), \end{aligned} \tag{11}$$

Here, the operator $\mathrm{L} = \frac{d}{dt}$. The initial values we suggest are given in the following,

$S_o(t) = S(0), \quad I_o(t) = I(0), \qquad R_o(t) = R(0)$ (12)

Let us assume the solution of (11) in the form,

$$\begin{aligned} S^*(t) &= S_o^*(t) + qS_1^*(t) + q^2S_2^*(t) + ... \\ I^*(t) &= I_0^*(t) + qI_1^*(t) + q^2I_2^*(t) + ... \\ R^*(t) &= R_0^*(t) + qR_1^*(t) + q^2R_2^*(t) + ... \end{aligned} \tag{13}$$

The use of (13) in (11) and collecting the same power we obtain,

$$\begin{aligned} &\mathrm{L}S(t) - \mathrm{L}S^{o}(t) = 0, \\ &\mathrm{L}I(t) - \mathrm{L}I^{o}(t) = 0, \\ &\mathrm{L}R(t) - \mathrm{L}R^{o}(t) = 0, \end{aligned} \tag{14}$$

And

$$\begin{aligned} &\mathrm{L}S_1^*(t) - \mathrm{L}S^{o}(t) = \left(rS_o^*(1-\frac{S_o^*}{k}) - \frac{\beta S_o^*(t)I_o^*(t)}{1+\alpha S_o^*} - \mathrm{L}S_o^*(t)\right), \\ &\mathrm{L}I_o^*(t) - \mathrm{L}I^{*}o(t) = \left(\frac{\beta S_o^*(t)I_o^*(t)}{1+\alpha S_o^*} - (\mu_1+\gamma)I_o^*(t) - \mathrm{L}I_o^*(t)\right), \\ &\mathrm{L}R_o^*(t) - \mathrm{L}R_o^*(t) = \left(\gamma I_o^*(t) - \mu_2 R_o^*(t) - \mathrm{L}R_o^*(t)\right), \end{aligned} \tag{15}$$

with the conditions

$S_1^*(t) = 0, \quad I_1^*(t) = 0, \quad R_1^*(t) = 0.$ (16)

And

$$\begin{aligned} &\mathrm{L}S_2^*(t) = \left(rS_1^*(1-\frac{S_1^*}{k}) - \frac{\beta(S_o^*(t)I_1^*(t) + S_1^*(t)I_o^*(t))}{\alpha S_1^*}\right), \\ &\mathrm{L}I_2^*(t) = \left(\frac{\beta(S_1^*(t)I_o^*(t) + S_o^*(t)I_1^*(t))}{\alpha S_1^*} - (\mu_1+\gamma)I_1^*(t)\right), \\ &\mathrm{L}R_2^*(t) = \left(\gamma I_1^*(t) - \mu_2 R_1^*(t)\right), \end{aligned} \tag{17}$$

with the conditions

$S_2^*(t) = 0, \qquad E_2^*(t) = 0, \qquad I_2^*(t) = 0, \qquad V_2^*(t) = 0.$ (18)

Similarly, we obtain

$$\begin{aligned}
\mathrm{L}S_3^*(t) &= \Big(rS_2^*(1-\frac{S_2^*}{k}) - \frac{\beta(S_2^*(t)I_o^*(t)+S_1^*(t)I_1^*(t)+S_o^*(t)I_2^*(t))}{\alpha S_2^*}\Big),\\
\mathrm{L}I_2^*(t) &= \Big(\frac{\beta\beta(S_2^*(t)I_o^*(t)+S_1^*(t)I_1^*(t)+S_o^*(t)I_2^*(t))}{\alpha S_2^*} - (\mu_1+\gamma)I_2^*(t)\Big), \qquad (19)\\
\mathrm{L}R_3^*(t) &= \Big(\gamma I_2^*(t) - \mu_2 R_2^*(t)\Big),
\end{aligned}$$

To get the solution, consider q=1 in (11), we obtain

$$\begin{aligned}
S^*(t) &= S_o^*(t)+S_1^*(t)+S_2^*(t)+...\\
I^*(t) &= I_0^*(t)+I_1^*(t)+I_2^*(t)+.... \qquad (20)\\
R^*(t) &= R_0^*(t)+R_1^*(t)+R_2^*(t)+...
\end{aligned}$$

Due to the rapid convergence of HPM, only a few iterations are enough for both linear and non-linear problems.

3.1 ZEROTH ORDER SOLUTION OR p^0

$$S_0^*(t)=130, \qquad E_0^*(t)=80, \quad I_0^*(t)=100, \qquad V_0^*(t)=220. \qquad (21)$$

3.2 FIRSTORDER SOLUTION OR p^1

$$\begin{aligned}
\mathrm{L}S_1^*(t) &= \Big(ru_1(1-\frac{u_1}{k}) - \frac{\beta u_1 u_2}{1+\alpha u_1}\Big)t,\\
\mathrm{L}I_1^*(t) &= \Big(\frac{\beta u_1 u_2}{1+\alpha u_1} - (\mu_1+\gamma)u_2\Big)t,\\
\mathrm{L}R_1^*(t) &= \Big(\gamma u_1 - \mu_2 u_3\Big)t,
\end{aligned}$$

And, we choose,

$$S_0^*(t)=130=u_1, \qquad I_0^*(t)=80=u_2, \qquad R_0^*(t)=100=u_3.$$

3.3SECOND ORDER SOLUTION OR P^2

$$\begin{aligned}
S_2^*(t) &= r\Big(ru_1(1-\frac{u_1}{k}) - \frac{\beta u_1 u_2}{1+\alpha u_1}\Big)\frac{t^2}{2}\Big\{1 - \frac{\Big(ru_1(1-\frac{u_1}{k}) - \frac{\beta u_1 u_2}{1+\alpha u_1}\Big)}{k}\Big\}\frac{t^2}{2} - \beta\Big\{\frac{u_1\Big(\frac{\beta u_1 u_2}{1+\alpha u_1} - (\mu_1+\gamma)u_2\Big)\frac{t^2}{2} + u_2\Big(ru_1(1-\frac{u_1}{k}) - \frac{\beta u_1 u_2}{1+\alpha u_1}\Big)}{\alpha\Big(ru_1(1-\frac{u_1}{k}) - \frac{\beta u_1 u_2}{1+\alpha u_1}\Big)}\Big\}\frac{t^2}{2},\\
I_2^*(t) &= \beta\Big\{\frac{u_1\Big(\frac{\beta u_1 u_2}{1+\alpha u_1} - (\mu_1+\gamma)u_2\Big)\frac{t^2}{2} + u_2\Big(ru_1(1-\frac{u_1}{k}) - \frac{\beta u_1 u_2}{1+\alpha u_1}\Big)}{\alpha\Big(ru_1(1-\frac{u_1}{k}) - \frac{\beta u_1 u_2}{1+\alpha u_1}\Big)}\Big\}\frac{t^2}{2} - (\mu_1+\gamma)\Big(\frac{\beta u_1 u_2}{1+\alpha u_1} - (\mu_1+\gamma)u_2\Big)\frac{t^2}{2},\\
R_2^*(t) &= \gamma\Big(\frac{\beta u_1 u_2}{1+\alpha u_1} - (\mu_1+\gamma)u_2\Big)\frac{t^2}{2} - \mu_2\Big(\gamma u_1 - \mu_2 u_3\Big)\frac{t^2}{2}.
\end{aligned}$$

4. NUMERICAL RESULTS AND DISCUSSION

In this section, we find the numerical simulation of the model (9) by using HPM, and the results are compared with others standards methods NSFD and RK4. The results obtained from HPM have good agreement with NSFD and RK4. The values of the parameters used in the numerical simulations are presented in Table 1. Figure1 represent the population of susceptible individuals. Figure2 represent the infected population and figure3 represent the population of recovered individuals. In this paper we solved an SIR model by homotopy perturbation method; first we model the problem and then applying the techniques of homotopy to find the analytical solution of the model. By applying the techniques of homotpy perturbation method, first we obtained the zero order problems, by comparing the coefficient and then substitute the zeroth order problem in the first order problem, we got the solution for first order and similarly we got the solution to second order problem. The models were solved by NSFD and RK4. By comparing the HPM solution with NSFD and RK4, we found that the HPM solution is have a good agreement with

other standards method of NSFD and RK4. For justification purpose the numerical results for HPM is obtained in figure1 to 3.

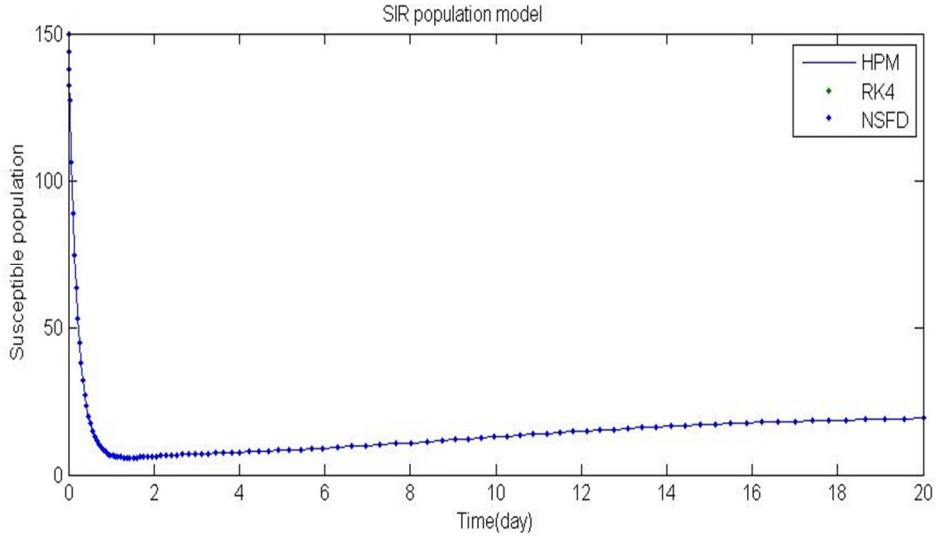

Figure 1: Represents the Susceptible human in the population.

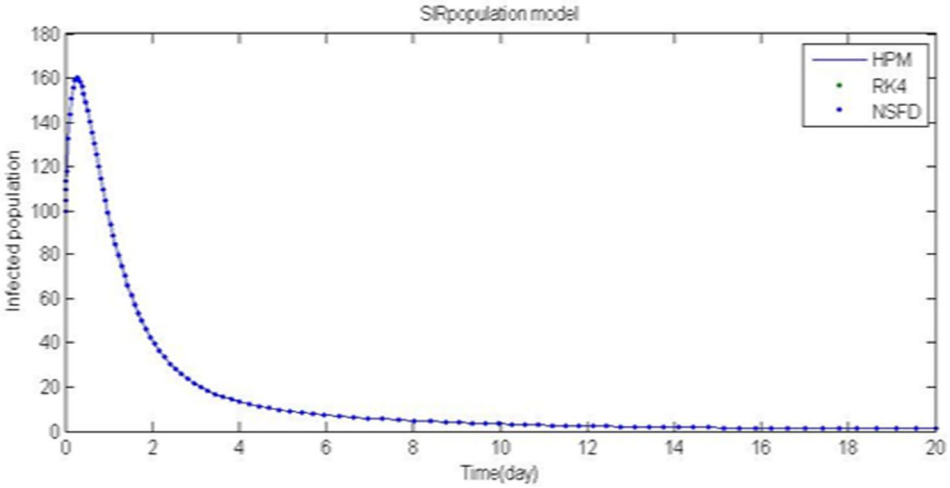

Figure 2: Represents the Infected human in the population

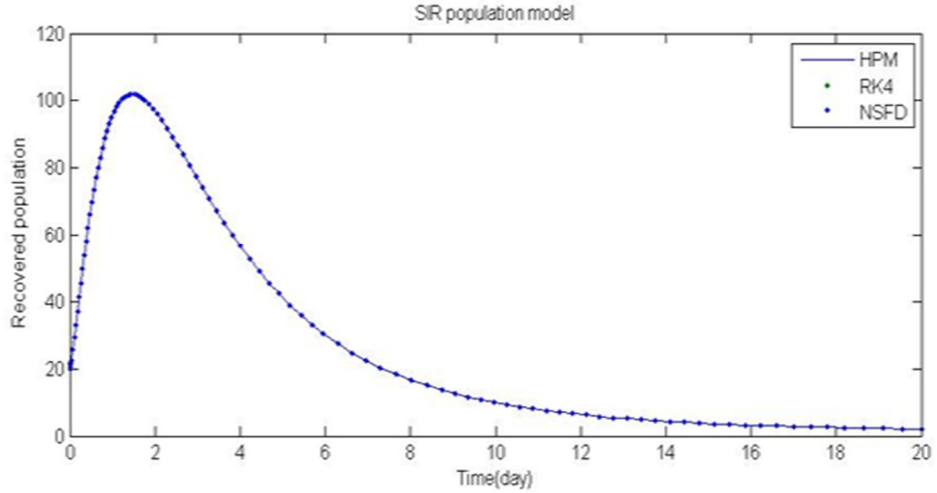

Figure 3: Represents the Recovered human in the population.

Table.1 The parameters and their values used in numerical simulations.

NOTATION	DESCRIPTION	VALUE
γ	The intrinsic growth rate of susceptible	0.3
κ	Carrying capacity of susceptible	0.78
α	The inhabilatory effect	0.05
β	The contact rate	0.3
μ_1	The death rate at infected class	0.0083
μ_2	The death rate at recovered class	0.0073

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