J. Appl. Environ. Biol. Sci., 7(12)27-34, 2017
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ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

Exact Solution of Tank Drainage through the Circular Pipe for Couple Stress Fluid

K. N. Memon[1,3*], A. M. Siddiqui[2], Syed Feroz Shah[1]

[1]Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan
[2] Pennsylvania State University, York Campus, Edgecombe 17403, USA
[3] Department of Mathematics and Statistics, QUEST, Nawabshah, Pakistan

Received: August 17, 2017
Accepted: October 12, 2017

ABSTRACT

This paper explores the problem of tank drainage through circular pipe of an unsteady, incompressible, isothermal couple stress fluid. The exact result is obtained from governing continuity and momentum equation's focus to proper preconditions. The Newtonian solution is retrieved from this proposed model on substitution $\gamma \to \infty$. Declaration on behalf of velocity profile, volume flux, average velocity, depth and also relationship how does the time vary with length and time required for complete drainage are obtained. Effects of various emerging parameters on velocity profile v_z and depth $H(t)$ are presented graphically.
KEYWORDS: Tank drainage, couple stress fluid, exact solution.

INTRODUCTION

In current years, non-Newtonian fluids have gained considerable attention because of their numerous biological, industrial and technological applications [16-19]. Here few cases of non-Newtonian fluids such as tooth paste, drilling mud, greases, paints, blood, polymer melts, clay coatings etc. It is an expansive class of fluids so; there is not a single model that can handle all the properties of such fluids as is done by the Newtonian fluids (described by the well-known Navier-Stokes equation) [20-23].
In this regard, several constitutive equations have been proposed to predict the physical structure and behavior of such fluids for different materials [6-7]. Between these, the couple stress fluid model proposed by V. K. Stokes in 1966 [8] has different characteristics, such as non-symmetric stress tensor, body couples and the presence of couple stresses. The couple stress theory was developed for particular fluids whose microstructure is mechanically momentous. The first theoretical study in cylindrical co-ordinates is given by K. C. Valanis and C. T. Sun [5], they also compare theoretical study with experimentally. Couple stress fluids are capable of portraying distinctive types of suspension fluids, blood, lubricants and so forth and contain lot of application's, especially in industry like a extrusion of polymer fluids, cooling of metallic plate in a bath, colloidal solutions and solidification of liquid crystals etc. Extensive study about couple stress fluid dynamics has additionally written by Stokes in 1984 [9]

The drainage of a fluid through pipe of a tank under the action of gravity is an old, howevercomplicated problem. The tank may be drained by an attach pipe or may be drained throughevenhanded hole "orifice situation". The pipe possibly could be horizental or vertical or may contain a complete piping system with horizental extension and vertical drop with fittings and valve, etc. Usually tank has a shape of cylinderical contain a vertical wallhoweverbottom maybe conical hemisherical or by flat or might be additional shape. There is sometimesintrest in draining the tank should be totally dry in which situation the bottom shape needs to be accounted for and occasionally not.

Classifications of gravity draining fluid's are used extensively throughout industries, a small number of such classifications are: draining condensate into storage, water distribution, waste water management and dams, retrieval of chemicals from tank farm. The generated model will accurately represent tank draining behavior for all tanks with a similar setup. End effects, accuracy of time measurement, accuracy of height measurements and friction losses will be taken into consideration [2].
An outstanding evaluation of exact solutions of the "Navier-Stokes equation" has been given by Wang [3]. In this manuscript, we studied tank drainage problem of couple stress fluid through by cylindrical pipe. Exact solutions of the consequential differential equations subject to boundary conditions, are obtained. For taking parameter $\gamma \to \infty$,

***Corresponding Author:** K. N. Memon, Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan. Email:kamrannazir025@yahoo.com

we retrieve velocity profile for linearly viscous case [4]. Also relationships for velocity-profile, flow rate, average-velocity depth of fluid in the tank and time required for complete drainage are calculated.
Kashif [11] analyzed second grade fluid for preamble effects over an oscillating plate and found expression of velocity field and shear stresses using Laplace and Fourier sine transforms on the governing partial differential equations of fluid flows. Kashif et al. [12] worked on Rayleigh stokes problem on generalized burger fluid using finite Fourier sine and Laplace transforms. They found velocity field using limiting cases of Newtonian and non-Newtonian fluids for different models. In another study, the effects of magnetic field on fractionalized viscoelastic fluid have been observed by Kashif et al. [13]. They found that behavior of fluid flow in presence of magnetic field was resistive due to Lorentz force. Kamran et al. [14] worked on un steady drainage problem using power law model down a vertical cylinder by using Jeffrey's approach. Kashif et al. [15] also investigated the impact of uniform and non-uniform magnetic field on Maxwell fluid.
This paper is organized by means of follows: Section number 2 provides basic equation's for the couple stress fluid. Section number 3 provides formulation and solution of the problem. Section number 4 deals with volume flux, average velocity, relationship how does the time vary with length and time required for complete drainage. Results and discussion are given in section number 5, while conclusion is provided in section number 6.

2 Basic Equations

Essential governing equations for incompressible couple stress fluid disregarding thermal effects are [10,24]:

$$\nabla \cdot \mathbf{V} = 0. \quad (1)$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{b} + \nabla \cdot \mathbf{T} - \eta \nabla^4 \mathbf{V}, \quad (2)$$

The symbol $\mathbf{V}$ represent velocity vector, ρ denotesthe constant density, p be the dynamic pressure, $\mathbf{b}$ is the body force, $\mathbf{T}$ the extra stress tensor and η is the couple stress parameter. The operator $\frac{D}{Dt}$ denotes the material derivative. The extra stress tensor describing a Newtonian fluid is made by:

$$\mathbf{T} = \mu \mathbf{A}_1. \quad (3)$$

Here μ represent is the coefficient of viscosity and $\mathbf{A_1}$ be the 1st Rivlin Ericksen tensor,represented as:

$$\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T.$$

3 Tank drainage

Consider a cylindrical tank containing an incompressible couple stress fluid. The radius of the tank is assumed to be R_T and diameter d. The initial depth of the fluid is chosen to be H_0. The fluid in the tank is drained down by means of a pipe having radius R and length L. Further more letting $H(t)$ be the depth of fluid in the tank at any time t.Flow of fluid in the pipe is due to gravity and pressure of the fluid in the tank.

We plane to calculate the velocity profile, pressure profile, flow rate, average velocity, relationship how does the time vary with length and the time required for complete drainage. Here we use cylindrical coordinates (r, θ, z) with r *-axis* normal to the pipe and z *-axis* along the center of the pipe in vertical direction. As the flow is individual in the z -direction and the θ and r components of velocity vector $\mathbf{V}$ are equal to zero,

$$V = [v_r, v_\theta, v_z] = [0, 0, v_z(r,t)]. \quad (4)$$

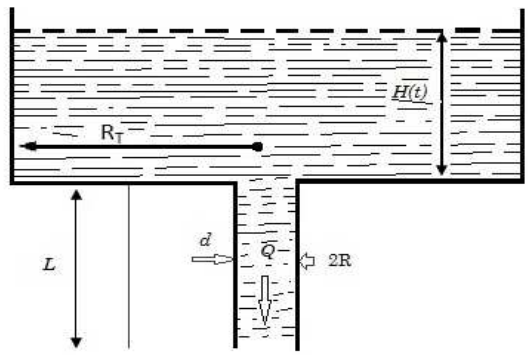

Figure 1: Geometry of the tank drainage flow down through pipe

Using profile (4), the equation of continuity (1) is indistinguishably fulfilled and the momentum equation (2) diminishes toward

$$r-component\ of\ momentum:\quad \frac{\partial p}{\partial r}=0, \tag{5}$$

$$\theta-component\ of\ momentum:\quad \frac{1}{r}\frac{\partial p}{\partial \theta}=0, \tag{6}$$

$$z-component\ of\ momentum:\quad \rho\frac{\partial v_z}{\partial t}=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z-\eta\nabla^4 v_z+\rho g. \tag{7}$$

From equations (5 - 7) we can see that the equation of motion is now quite simple, yielding that the pressure is only function of z and t and the equation to be solved for $v_z(r,t)$ is

$$\rho\frac{\partial v_z}{\partial t}=-\frac{\partial p}{\partial z}+\mu\nabla^2 v_z-\eta\nabla^4 v_z+\rho g. \tag{8}$$

Equation (8) is a partial differential equation for p and v_z. The velocity in the pipe flow remains nearly constant with time due to slow draining such that we may neglect the time derivative $\frac{\partial v_z}{\partial t}$. Also flow in the pipe of radius R is due to both gravity and hydrostatic pressure. The pressures at the pipe entrance and exit are respectively,

$$at\quad z=0,\quad p=p_1=\rho gH(t),$$
$$at\quad z=L,\quad p=p_2=0,$$

so that

$$\frac{\partial p}{\partial z}=-\frac{\rho gH(t)}{L} \tag{9}$$

The equation of motion (8) thus reduces to

$$\nabla^2(\mu-\eta\nabla^2)v_z=-\rho g\left[\frac{H(t)}{L}+1\right]. \tag{10}$$

To solve equation (10) first we let

$$\mu v_z-\eta\nabla^2 v_z=\phi \tag{11}$$

By using equation (11) then we can write equation (10) as:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)=-\rho g\left[\frac{H(t)}{L}+1\right]. \tag{12}$$

Associated boundary conditions are taken as [5]

$$\phi\quad finite\quad at\quad r=0 \tag{13}$$

$$v_z\quad finite\quad at\quad r=0 \tag{14}$$

$$v_z = 0 \qquad at \qquad r = R \tag{15}$$

$$\frac{d^2 v_z}{dr^2} - \eta' \frac{1}{R}\frac{dv_z}{dr} = 0 \qquad at \qquad r = R \tag{16}$$

Where η' is constant associated with couple stress, solving equation (12) subject to the boundary condition (13), we get

$$\frac{d\phi}{dr} = -\frac{\rho g r}{2}\left[\frac{H(t)}{L} + 1\right]. \tag{17}$$

Integrate to equation number (17) with respect to r then substitute into equation (11), once we obtain

$$\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr} - \gamma^2 v_z = \frac{\rho g}{\eta}\left[\frac{H(t)}{L} + 1\right]\left(\frac{r^2}{4} + C_1\right) \tag{18}$$

Where C_1 represents the constant of integration and $\gamma^2 = \frac{\mu}{\eta}$. Complementary integral of equation (18) taking account of act from equation (14) is given by

$$v_z = C_2 I_0(\gamma r) \tag{19}$$

Here I_0 is the modified Bessel function of order zero and C_2 is the constant of integration. Particular integral of equation (18) is given by direct substitution.

$$v_z = -\frac{\rho g}{\gamma^2 \eta}\left[\frac{H(t)}{L} + 1\right]\left(\frac{r^2}{4} + C_1\right) \tag{20}$$

Thus the complete solution of v_z can be written as

$$v_z = C_2 I_0(\gamma r) - \frac{\rho g}{\gamma^2 \eta}\left[\frac{H(t)}{L} + 1\right]\left(\frac{r^2}{4} + C_1\right) \tag{21}$$

By using boundary conditions from equation (15) and (16) into equation (21) for evaluating the values of constant of integration, after considerable calculation once we get

$$v_z = \frac{\rho g}{4\mu}\left[\frac{H(t)}{L} + 1\right]\left(R^2 - r^2\right) - \frac{\rho g\left[\frac{H(t)}{L} + 1\right](\eta - \eta')\left[I_0(\gamma R) - I_0(\gamma r)\right]}{2\gamma\mu\left[\gamma\eta I_0(\gamma R) - (\eta + \eta')\left(\frac{I_1(\gamma R)}{R}\right)\right]} \tag{22}$$

Where I_1 is the modified Bessel function of order one and Note that for $\gamma \to \infty$, we recover The solution for Newtonian fluid [4] which is first term of right hand side

Flow rate, average velocity and relation of depth of the tank with respect to time

The "flow rate Q'' per unit width is specified through the formula

$$Q = \int_0^R 2\pi r v_z(r,t)\, dr. \tag{23}$$

Using velocity profile (22) in equation (23), one can calculate the flow rate

$$Q = \frac{\rho g\left[H(t) + L\right]\pi R^3\left[\left\{\eta\left(-4 + R^2\gamma^2\right) + 4\eta'\right\} I_0(R\gamma) - (\eta + \eta')R\gamma I_1(R\gamma) + 4(\eta - \eta')_0 \widetilde{F}_1\left(;2;\frac{R^2\gamma^2}{4}\right)\right]}{8L\gamma\mu\left(R\gamma\,\eta\, I_0(R\gamma) - (\eta + \eta')R\gamma\, I_1(R\gamma)\right)} \tag{24}$$

Here ${}_0\widetilde{F}_1\left(;2;\frac{R^2\gamma^2}{4}\right)$ represent Hypergeometric0F1regularized function, which can be specified as:

$$ {}_0\widetilde{F}_1\left(;2;\frac{R^2\gamma^2}{4}\right)=\sum_{k=0}^{\infty}\left(\frac{R^2\gamma^2}{4}\right)^k\frac{1}{\Gamma(2+k)k!} \tag{25}$$

We determine the average velocity, $\bar{v}$ by using the formula

$$\bar{V}=\frac{Q}{\pi R^2}. \tag{26}$$

So the average velocity of the fluid flowing down the pipe is

$$\bar{V}=\frac{\rho g[H(t)+L]R\left[\left\{\eta\left(-4+R^2\gamma^2\right)+4\eta'\right\}I_0(R\gamma)-(\eta+\eta')R\gamma I_1(R\gamma)+4(\eta-\eta')_0\widetilde{F}_1\left(;2;\frac{R^2\gamma^2}{4}\right)\right]}{8L\gamma\mu\left(R\gamma\eta I_0(R\gamma)-(\eta+\eta')R\gamma I_1(R\gamma)\right)} \tag{27}$$

Mass balance over the entire tank is

$$\frac{d}{dt}\left[\pi R_T^2 H(t)\right]=-Q(t). \tag{28}$$

Substituting flow rate from equation (24) into equation (28) and then separating variables on both sides of equation one obtains

$$H(t)=\left(H_0+L\right)e^{-\frac{\rho g R^3 t\left[\left\{\eta\left(-4+R^2\gamma^2\right)+4\eta'\right\}I_0(R\gamma)-(\eta+\eta')R\gamma I_1(R\gamma)+4(\eta-\eta')_0\widetilde{F}_1\left(;2;\frac{R^2\gamma^2}{4}\right)\right]}{8R_T^2L\gamma\mu\left(R\gamma\eta I_0(R\gamma)-(\eta+\eta')R\gamma I_1(R\gamma)\right)}}-L, \tag{29}$$

and the time required for complete drainage is obtained by taking $H(t)=0$ in

$$t=\frac{-8R_T^2L\gamma\mu\left(R\gamma\eta I_0(R\gamma)-(\eta+\eta')R\gamma I_1(R\gamma)\right)}{\rho g R^3 t\left[\left\{\eta\left(-4+R^2\gamma^2\right)+4\eta'\right\}I_0(R\gamma)-(\eta+\eta')R\gamma I_1(R\gamma)+4(\eta-\eta')_0\widetilde{F}_1\left(;2;\frac{R^2\gamma^2}{4}\right)\right]}\ln\left(\frac{H(t)+L}{H_0+L}\right). \tag{30}$$

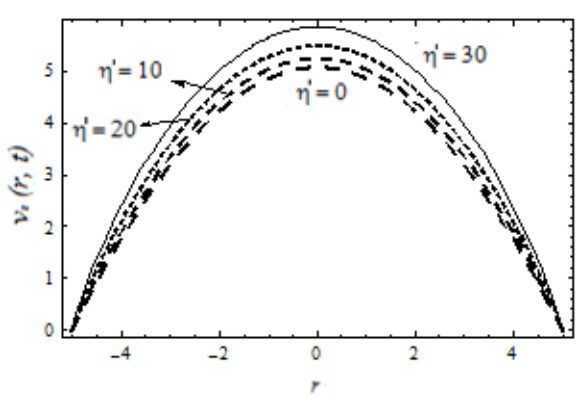

Figure 2: Effect of η' on velocity profile, when
$\mu=27.5.5\,poise, \rho=0.78\,g/cm^3$
$R=5cm, L=10cm, H(t)=20cm., \eta=8$

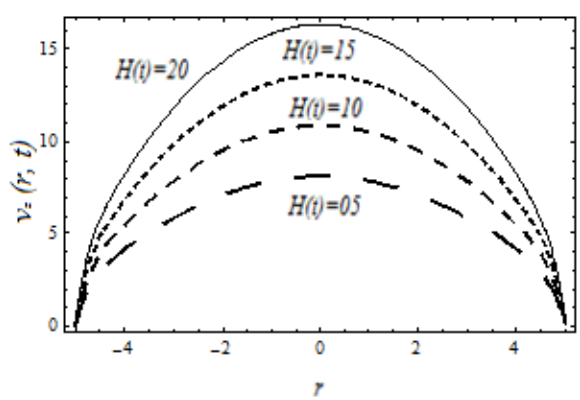

Figure 3: Effect of $H(t)$ on velocity profile, when
$\mu=11.5\,poise, \rho=0.78\,g/cm^3$
$R=5\,cm, L=10\,cm, \eta=0.6, \eta'=10$

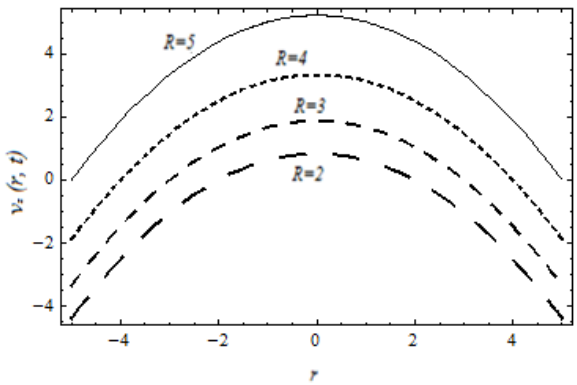

Figure 4: Effect of R on velocity profile, when
$\mu = 27.5\,poise, \rho = 0.78\,g/cm^3$
$L = 10\,cm$, $H(t) = 20\,cm$, $\eta = \eta' = 1$.

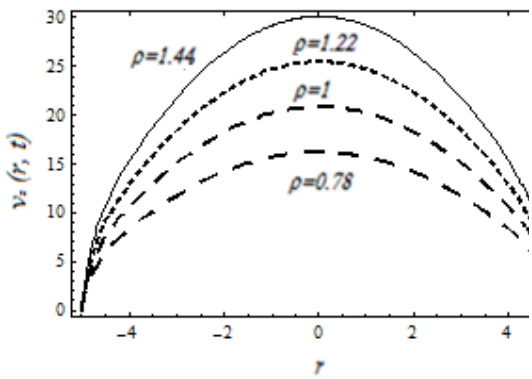

Figure 5: Effect of ρ on velocity profile, when
$\mu = 11.5\,poise, R = 5cm$
$L = 10cm, H(t) = 20cm, \eta = 0.6, \eta' = 10.$

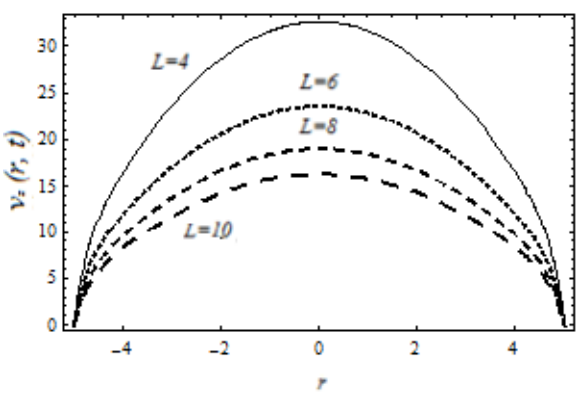

Figure 6: Effect of L on velocity profile, when
$\mu = 11.5\,poise, \rho = 0.78\,g/cm^3$
$R = 5cm$, $H(t) = 20\,cm$, $\eta = 0.6, \eta' = 10$.

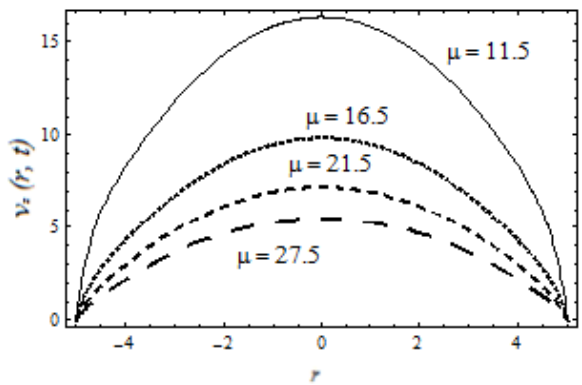

Figure 7: Effect of μ on velocity profile, when
$\beta = 0.01, \rho = 0.78\,g/cm^3$
$R = 5cm, L = 10cm, H(t) = 20cm.$

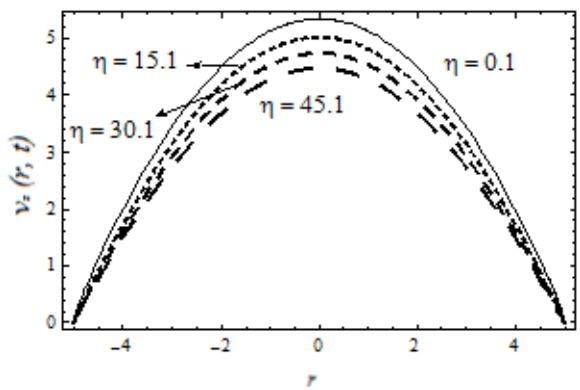

Figure 8: Effect of η on velocity profile, when
$\eta = 31.5\,poise, \rho = 0.78\,g/cm^3, L = 10cm,$

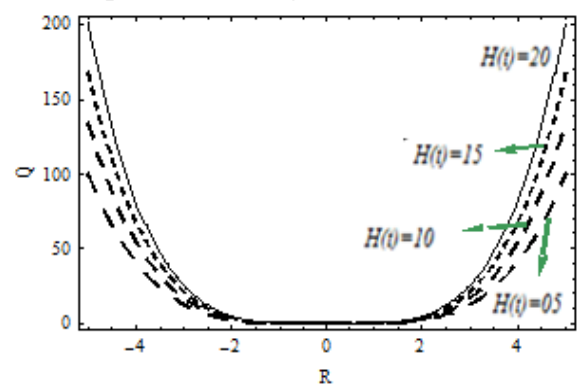

Figure 9: Effect of $H(t)$ on flow rate, when
$\mu = 27.5\,poise, \rho = 0.78\,g/cm^3$
$R = 5\,cm$, $L = 10\,cm$, $\eta = 4$, $\eta' = 1$.

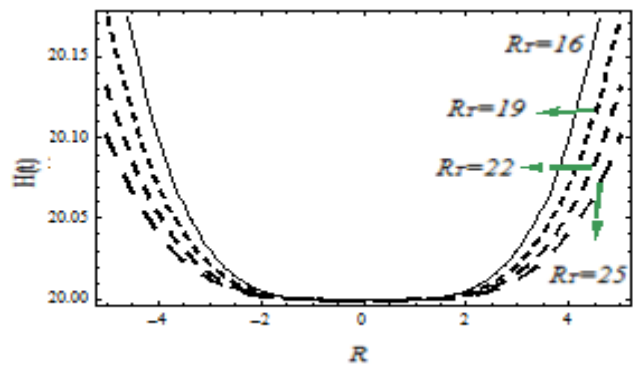

Figure10: Effect of R_T on depth, when

$$\mu = 27.5 poise, \rho = 0.78 g/cm^3, \eta = 6$$
$$t = 1, H_0 = 20cm, L = 10cm, \eta' = 1.$$

4 RESULTS AND DISCUSSION

In the above sections we studied tank drainage problem using an incompressible couple stress fluid, exact solutions for the differential equation is obtained by using Bessel and Hypergeometric0F1regularized function. The variation of velocity profile v_z, flow rate Q and depth $H(t)$ has been investigated on different parameters. The effects of the couple stress parameters $\eta' and\ \eta$, dynamic viscosity μ, depth $H(t)$, length of pipe L, pipe radius R and density ρ on velocity profile are observed through figures (2) - (8)and effect of the depth $H(t)$ on flow rate is shown in figure (9)and effect of the radius of tank R_T on depth $H(t)$ is examined in figure (10). In figures(2) – (8) it is detected that the magnitude of velocity increases as the increase with couple stress parameter η', depth $H(t)$, pipe radius R and density ρ and decreases for the increase of length of pipe L, dynamic viscosity μ and couple stress parameter η .In figure 9for the increase $H(t)$ we detected that flow rate increasesand in figure (10) depth $H(t)$ with respect to pipe radius R is plotted, we detect that $H(t)$ decrease with increase of radius of tank R_T.

5 Conclusions

Considering equation for unsteady, incompressible, isothermal tank drainage flow for the couple stress fluid. We have obtained exact solutions for "velocity profile, flow rate, average velocity and time required to complete drainage". Here itis noted that for the couple stress parameter $\gamma \to \infty$, solution (22) reduces to the Newtonian solution [4]. A relationship (30), how does the time vary with length is derived. It is noted that as the fluid is becoming thicker, velocity of the fluid decreases.

REFERENCES

[1] Derek, C; Tretheway, D. C; Meinhart, C. D Apparent fluid slip at hydrophobic micro channels walls, Physics of Fluid, 14; (2002), pp- L9-L12.
[2] Joe Leonared, S.T. Macklin, Jennifer Ogunyomi, Tank drainage modeling, Oklahoma State University, School of Chemical Engineering, Unit operation laboratory 3/25/09
[3] C. Y. Wang: Exact solutions of the steady-state Navier-Stockes equations, Annu. Rev. Fluid Mech. 23(1991), 159-177.
[4] T. C. Papanastasiou, Applied Fluid Mechanics, P T R Prentice Hall.
[5] K. C. Valanis, C. T. Sun, Poiseuille flow of a fluid with couple stress with application to blood flow, Biorheology, 1969, Vol. 6, pp. 85-87, Pergamon Press, Printed in Great Britain.

[6] N. S. Deshpande, M. Barigou, Vibrational flow of non-Newtonian fluids, Chemical Engineering Science, 56, 3845-3853 (2001).
[7] M. Kemiha, X. Frank, S. Poncin, H. Z. Li, Origin of the negative wake behind a bubble rising in non-Newtonian fluids, Chemical Engineering Science, **61**, 4041-4047 (2006).
[8] Stokes, V.K., 1966. Couple stresses in fluid. Physics of Fluids 9, 1709–1715.
[9] Stokes, V.K., 1984. Theories of Fluids with Microstructure: An Introduction. Springer-Verlag, New York.
[10] M. Farooq, M.T. Rahim, S. Islam, A.M. Siddiqui, Steady Poiseuille flow and heat transfer of couple stress fluids between two parallel inclined plates with variable viscosity, Journal of the Association of Arab Universities for Basic and Applied Sciences (2013) 14, 9–18.

[11] Kashif Ali Abro, Porous Effects on Second Grade Fluid in Oscillating Plate, Journal of Applied Environmental and Biological Sciences (JAEBS), 6(5) 71-82 (2016).

[12] Kashif Ali Abro, Mukarrum Hussain, Mirza Mahmood Baig, Khalil-ur-Rehman Channa, Analysis of Generalized Burger's Fluid in Rayleigh Stokes Problem, Journal of Applied Environmental and Biological Sciences (JAEBS), 7(5) 55-63 (2017).

[13] Kashif Ali Abro, Mukarrum Hussain, Mirza Mahmood Baig, Impacts of Magnetic Field on Fractionalized Viscoelastic Fluid, Journal of Applied Environmental and Biological Sciences (JAEBS), 6(9) 84-93 (2016).

[14] K.N. Memon, A.M. Siddiqi, Sayed Feroz Shah, Salman Ahmed, Unteady Drainage of the Power Law Fluid Model down a Vertical Clynderical, Journal of Applied Environmental and Biological Sciences (JAEBS), 4(9) 309-319 (2014).

[15] Kashif Ali Abro, Ambreen Siyal, Sher Khan Awan, Abdul Saleem Memon, Investigation of viscoelastic fluid with uniform and non-uniform magnetic source, Journal of Applied Environmental and Biological Sciences (JAEBS), (2017) (in press).

[16] Muzaffar Hussain Laghari, Kashif Ali Abro, Asif Ali Shaikh, Helical flows of fractional viscoelastic fluid in a circular pipe, International Journal of Advanced and Applied Sciences, 4(10) 97-105 (2017).

[17] Kashif Ali Abro, Ilyas Khan, Analysis of Heat and Mass Transfer in MHD Flow of Generalized Casson Fluid in a Porous Space Via Non-Integer Order Derivative without Singular Kernel, Chinese Journal of Physics, 55(4) 1583-1595(2017).

[18] Arshad Khan, Kashif Ali Abro, Asifa Tassaddiq, Ilyas Khan, Atangana-Baleanu and Caputo Fabrizio Analysis of Fractional Derivatives for Heat and Mass Transfer of Second Grade Fluids over a Vertical Plate: A Comparative study, Entropy, 19(8) 1-12, (2017).

[19] Kashif Ali Abro, Mukarrum Hussain, Mirza Mahmood Baig, Analytical Solution of MHD Generalized Burger's Fluid Embedded with Porosity, International Journal of Advanced and Applied Sciences, 4(7) 80-89, (2017).

[20] Shakeel Ahmed Kamboh, Zubair Ahmed Kalhoro, Kashif Ali Abro, Jane Labadin, Simulating Electrohydrodynamic Ion-Drag Pumping on Distributed Parallel Computing Systems, Indian Journal of Science and Technology, 10(24) 1-5(2017).

[21] Kashif Ali Abro, Muhammad Anwar Solangi, Heat Transfer in Magnetohydrodynamic Second Grade Fluid with Porous Impacts using Caputo-Fabrizoi Fractional Derivatives, Punjab University Journal of Mathematics, 49(2) 113-125 (2017).

[22] Kashif Ali Abro, Asif Ali Shaikh and Sanuallah Dehraj, Exact Solutions on the Oscillating Plate of Maxwell Fluids, Mehran University Research Journal of Engineering & Technology, 35(1) 157-162 (2016).

[23] Kashif Ali Abro, Muhammad Anwar Solangi, Muzaffar Hussain Laghari, Influence of Slippage in Heat and Mass Transfer for Fractionalized MHD Flows in Porous Medium, International Journal of Advances in Applied Mathematics and Mechanics, 4(4) 5-14(2017).

[24] K. C. Valanis and C.T. Sun, Poiseuille flow of a fluid with couple stress with applications to blood flow, Biotechnology, 6, 85-97, 1969.