Blending Two Parametric Quadratic Bezier Curves
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ABSTRACT

The main objective of this project is to show how to blend two parametric curves where the two parametric curves is blended to produce one new curve under certain conditions given. Normally, a Bezier curve is a parametric curve that is frequently used in Computer-Aided Design (CAD) and Computer-Aided Graphic Design (CAGD). Therefore, this kind of curve is selected to be used in this project that will focus on quadratic Bezier curve. Then, this idea on blending two parametric curves is expanded to blend the curves in plane. Furthermore, by this way, we can generate various shapes in three dimensional (3D).

KEYWORDS: Bezier curve, blending parametric curve.

INTRODUCTION

The concept and application of Bezier curves is used to model parametric curves and surfaces independently. Bezier curve is a base curve that is widely used in geometric design. There are many graphics packages which have been used Bezier curves in their CAGD system such as Adobe Illustrator, CorelDraw and generate fonts to PostScript. In 1998, Hui described that a curve can be blended into another curve. Farin (2002) stated that by blending bilinear curves can produce a patch Coons. In Piegl (1997), the term function is defined as mixture of basic function for the curve involved. The formula usually is created in an orderly and organized method so that the results of the curve which is blended qualify and correspond to each point on the curve bases. Many theories and methods have been introduced in the blending curves and surfaces involving Hermite interpolation, cubic ratio interpolation splints and Casteljau algorithm. There are several methods has been used to solve problems involving blending curves. For example, C.Hoffmann and J.Hopcroft (1986) suggest a potential method for blending surface automatically, but they provided that the surface to be blended should be quadric. Therefore, Jinsan Cheng (2002) discussed the continuity and smoothness in combining these two quadric surfaces in explicit formula. A curved base is built to connect the two axes of the surface to be blended. Through this method, the surface needs a normal vector at each point within the limits.

Bezier curve is defined by the endpoints of the curve and control points that will determine the degree of curvature. This curve does not provide a local control point. So, by changing the control points it will affect the overall shape of the curve. This project shows how two quadratic Bezier curves with parameter \( t \) are blended or mixed together to produce a new third curve according to certain conditions. In addition, this project also discusses the blending of free surface equation.

MATERIAL AND METHOD

Bezier Curve

Bezier curve with degree \( n \) is known as

\[
B(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i P_i
\]

\[
= (1-t)^n P_0 + \binom{n}{1}(1-t)^{n-1}tp_1 + \cdots + \binom{n}{n-1}(1-t)t^{n-1}P_{n-1} + t^n P_n
\]

where \( t \) is a parameter, \( t \in [0,1] \) and \( \binom{n}{i} \) is a binomial coefficient. \( P_0, P_1, P_2, \ldots, P_n \) are the control points.
Quadratic Bezier Curve

Given that three control points $P_0, P_1$ dan $P_2$, we divide the segment based on parameter $t$ where $t \in [0,1]$.

- Let $P_1^{(1)}$ is a point for segment $P_0P_1$ by using the following equation:
  \[ P_1^{(1)} = (1-t)P_0 + tP_1 \] (1.1)

- Let $P_2^{(1)}$ is a point for segment $P_1P_2$ by using the following equation
  \[ P_2^{(1)} = (1-t)P_1 + tP_2 \] (1.2)

- Let $P_2^{(2)}$ is a point for segment $P_1^{(1)}P_2^{(1)}$ by using the following equation
  \[ P_2^{(2)} = (1-t)P_1^{(1)} + tP_2^{(1)} \] (1.3)

- Given
  \[ P(t) = P_2^{(2)} \]

Based on the three equations above, each point $P_1^{(1)}, P_2^{(1)}$ dan $P_2^{(2)}$ is a function with parameter $t$. $P_2^{(2)}$ can be solved using $P(t)$. Therefore, $P(t)$ is determined as a function of Bezier curve.

\[ P(t) = P_2^{(2)} = (1-t)P_1^{(1)} + tP_2^{(1)} \] (1.4)

Equation (1.1) and (1.2) then substituted into equation (1.4), we get

\[ P(t) = (1-t)^2P_0 + 2(1-t)tP_1 + t^2P_2, \quad t \in [0,1] \] (1.5)

Equation (1.5) is a quadratic polynomial with parabolic shaped. Therefore, quadratic Bezier curve is a parabolic curve.

Figure 1
Quadratic Bezier curve
Blending curves
Let \( k_1 \) and \( k_2 \) are curves with parameter \( t \) where \( t \in [0,1] \). \( b(t) \) is required to be start with \( t_0 = 0 \) at point \( k_1(0) = A_1 \) for the curve \( k_1 \) and last at \( t_1 = 1 \) at the point \( k_2(1) = B_2 \) for the curve \( k_2 \). Therefore,

\[
\begin{align*}
b(0) &= k_1(0) = A_1, & k_1(1) &= A_2 \\
b(1) &= k_2(1) = B_2, & k_2(0) &= B_1
\end{align*}
\]

In addition, the curve \( b(t) \) should also have tangent vectors which are

\[
\begin{align*}
b'(0) &= k_1'(0) = D_0 \\
b'(1) &= k_2'(1) = D_e
\end{align*}
\]

The equation for the new curve by blending \( k_1 \) dan \( k_2 \) together is given by

\[
b(t) = [1 - H(t)]k_1(t) + H(t)k_2(t), \quad 0 \leq t \leq 1
\]

Based on the condition \( b(0) = k_1(0) \) and \( b(1) = k_2(1) \), it shows that \( H(0) = 0 \) and \( H(1) = 1 \). The lowest degree polynomial for \( H(t) \) is one. Thus we obtain

\[
b(t) = [1 - t]k_1(t) + tk_2(t), \quad 0 \leq t \leq 1
\]

Blending two quadratic Bezier curve
Quadratic Bezier curve is generally given by the following equation

\[
\begin{align*}
k_1(t) &= (1 - t)p_0 + 2t(1 - t)p_1 + t^2p_e \\
k_2(t) &= (1 - t)q_0 + 2t(1 - t)q_1 + t^2q_e
\end{align*}
\]

Equation (2.1) is defined as the following with \( t = 0 \) at start and end at \( t = 1 \).

\[
k_1(0) = p_0 = (x_i, y_i) \\
k_2(0) = q_0 = (x_i, y_i) \\
k_1(1) = p_e = (x_{i+1}, y_{i+1}) \quad \text{for } i = 0, 1, 2, ..., k \\
k_2(1) = q_e = (x_{i+1}, y_{i+1})
\]

For the curve \( b(t) \), it must satisfy

\[
\begin{align*}
b(0) &= k_1(0) = p_0 \\
b(1) &= k_2(1) = p_e
\end{align*}
\]

Therefore, equation (2.1) can be simplified as

\[
b(t) = (1 - t)((1 - t)p_0 + 2t(1 - t)p_1 + t^2p_e) \\
+ t((1 - t)^2q_0 + 2(1 - t)tq_1 + t^2q_e)
\]

Tangent vector for equation (2.1) and (2.2) is determined by differentiate those three equations \( k_1, k_2 \) and \( b \) with respect to \( t \) and we obtained

\[
\begin{align*}
k_1'(t) &= -2(1 - t)p_0 + 2(1 - t)p_1 - 2tp_1 + 2tp_e \\
k_2'(t) &= -2(1 - t)q_0 + 2(1 - t)tq_1 - 2tq_1 + 2tq_e \\
b'(t) &= -(1 - t)^2p_0 - 2(1 - t)t^2p_e + (1 - t)(-2(1 - t)p_0 \\
&\quad + 2(1 - t)p_1 - 2tp_1 + 2tp_e) + (1 - t)^2q_0 + 2(1 - t)tq_1 \\
&\quad + t^2q_e + t(-2(1 - t)q_0 + 2(1 - t)tq_1 - 2tq_1 + 2tq_e)
\end{align*}
\]
Then tangent vector for Equation (2.1) and (2.2) with \( t = 0 \) at start and end at \( t = 1 \) are as follows

\[
\begin{align*}
    k_1'(0) &= -2p_0 + 2p_1 \\
    k_2'(0) &= -2q_0 + 2q_1 \\
    k_1'(1) &= -2p_1 + 2p_e \\
    k_2'(1) &= -2q_1 + 2q_e \\
    b'(0) &= -3p_0 + 2p_1 + q_0 \\
    b'(1) &= -p_e - 2q_1 + 3q_e 
\end{align*}
\]  
(2.3)

From (2.3), we found that \( b'(0) \neq k_1'(0) \) and \( b'(1) \neq k_2'(1) \).

In order to qualify the requirement for curve \( b(t) \), the starting point and the end point for the two curves \( k_1 \) and \( k_2 \) should be the same. So \( k_1(0) = k_2(0) = p_0 \) and \( k_1(1) = k_2(1) = p_e \).

Equation (2.1) and (2.2) become

\[
\begin{align*}
    k_1(t) &= (1 - t)p_0 + 2t(1 - t)p_1 + t^2p_e \\
    k_2(t) &= (1 - t)p_0 + 2t(1 - t)q_1 + t^2p_e \\
    b(t) &= (1 - t)((1 - t)p_0 + 2t(1 - t)p_1 + t^2p_e) \\
    &\quad + t((1 - t)^2q_0 + 2(1 - t)q_1 + t^2q_e) 
\end{align*}
\]  
(2.4)

Then, the tangent vector for equation (2.4) with \( t = 0 \) at start and end at \( t = 1 \) are as follows

\[
\begin{align*}
    k_1'(0) &= -2p_0 + 2p_1 \\
    k_2'(0) &= -2p_0 + 2q_1 \\
    k_1'(1) &= -2p_1 + 2p_e \\
    k_2'(1) &= -2q_1 + 2p_e \\
    b'(0) &= -2p_0 + 2p_1 \\
    b'(1) &= -2p_0 + 2p_1 
\end{align*}
\]  
(2.5)

From (2.5), the tangent vector for \( b(t) \) follows the condition where

\[
\begin{align*}
    b'(0) &= k_1'(0) \\
    b'(1) &= k_2'(1)
\end{align*}
\]

From Figure 2, \( k_1 \) is plotted with coordinate \( p_0 = (2, 3), p_1 = (3, 2), p_e = (4, 4) \) and tangent vector \( D_0 = (2, -2) \) and \( D_1 = (2, 4) \).
Figure 3

$k_2$ is plotted with coordinate $p_0 = (2, 3), \; q_1 = (3, -2), \; p_e = (4, 4)$ and tangent vector $D_2 = (2, -10)$ and $D_e = (2, 12)$.

Figure 4

the new curve $b(t)$ formed by blending the curves $k_1$ and $k_2$.

Based on figure 4, it shows that the three curves $k_1, k_2$ and $b$ are plotted on the same axis. The curve $b$ that is generated follows the condition where it starts at the same point at $p_0 = (2, 3)$ and ends at the same point at $p_e = (4, 4)$. The vector tangent also meets the conditions where $b'(0) = k_1'(0) = (2, -2)$ and $b'(1) = k_2'(1) = (2, 12)$.

**Blending two quadratic Bezier curve in a plane**

According to the equation 2.1 $p_0, p_1, p_e, q_0, q_1$ and $q_e$ are points in plane $(x, y, z)$. For this discussion, curve $k_1$ is plotted in $z-axis$ so that the coordinate is $(x, y, 0)$. For the curve $k_2$, it is plotted in $x-axis$ so that the coordinate is $(0, y, z)$. Therefore, the control points are given by

\[
\begin{align*}
  p_0 &= (x_0, y_0, 0), & p_1 &= (x_1, y_1, 0), & p_e &= (x_2, y_2, 0) \\
  q_0 &= (0, y_0, z_0), & q_1 &= (0, y_1, z_1), & q_e &= (0, y_2, z_2)
\end{align*}
\]

We have

\[
\begin{align*}
  k_1 &= ((1 - t)^2 x_0 + 2(1 - t)tx_1 + t^2 x_2, (1 - t)^2 y_0 + 2(1 - t)ty_1 + t^2 y_2, 0) \\
  k_2 &= (0, (1 - t)^2 y_0 + 2(1 - t)ty_1 + t^2 y_2, (1 - t)^2 z_0 + 2(1 - t)tz_1 + t^2 z_2)
\end{align*}
\]
Based on Figure 5, \(k_1(t)\) is a parabolic curve formed along the \(z\)-plane. While \(k_2(t)\) is a parabolic curve formed along the \(x\)-plane.

**DISCUSSION**

**Linear Blending for Quadratic curve**

Given a linear function with the following general equation

\[
k_3(t) = (1 - t)r_0 + tr_e
\]

The three curves \(k_1, k_2\) and \(k_3\) will be blending together to form a new shaped based on the following surface equation.

\[
S(u) = (1 - u)^2k_1(t) + 2u(1 - u)k_3(t) + u^2k_2(t)
\]

where \(t \in (0,1)\) and \(u \in (0,1)\).

The coordinate for point \(r_0\) and \(r_e\) should be controlled to ensure that the result is as expected. The coordinate for point \(r_0\) and \(r_e\) are set as \(r_0 = (x_0, y_0, z_0)\) and \(r_e = (x_2, y_b, z_2)\). Therefore we have

\[
k_3 = (1 - t)x_0 + tx_2, (1 - t)y_0 + ty_b, (1 - t)z_0 + tz_2
\]

In a plane surface, \(k_3(t)\) is a straight line along \(y\)-plane.
Based on Figure 7 (a), when $y_a = y_b = 1$, the curve $k_3(t)$ intercepts the y-axis at $y = 1$. Figure 7 (b) shows the curve $k_3(t)$ intercepts the y-axis at $y = 2$. By setting $y_a = y_b = y_2$, $k_3(t)$ will be in the same plane with the curve $k_1(t)$ and $k_2(t)$. When the value of $y_a$ and $y_b$ are greater than $y_2$, it can be seen that the interception between the three curves form a new curve between $k_1(t)$ and $k_2(t)$ as in figure 7 (d) and (e).
Based on figure 8, the value of $y_a < y_b$. The curve $k_3(t)$ crosses the curves $k_1(t)$ and $k_2(t)$ at $y_a$ in the front and at $y_b$ at the behind.

**CONCLUSIONS**

This whole project discuss about the two parametric curves that are blend together to generate a new curve by using some specific conditions and rules. If two curves for Bezier functions are blended, the most important thing is to control the control points so that the new third curve is as expected. The more points that need to be controlled, it is difficult to produce the curved as desired. The advantage of using Bezier curve is it is easily formed by controlling the control points.

**REFERENCES**

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