J. Basic. Appl. Sci. Res., 3(6)516-521, 2013
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ISSN 2090-4304
Journal of Basic and Applied Scientific Research
www.textroad.com

α-Returns to Scale in Production Technology with Variable Returns to Scale

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ABSTRACT

In this paper, we discuss strictly increasing and strictly decreasing returns to scale, and then we introduce the homogeneous production technology and its relationship with α-returns to scale. The definition of α-returns to scale with the variable returns to scale assumption is then provided (the BCC model). After that, new assumptions and theorems are proposed and proved.

KEYWORDS: Data envelopment analysis, α-Returns to scale, Homogeneous technology, Variable returns to scale.

1. INTRODUCTION

One of the branches of operations research is data envelopment analysis (DEA), which is an effective method in assessment and analysis of system with multiple inputs and outputs.

Productivity growth results from technology change and technical efficiency profit. Macroeconomic productivity growth is measured usually without regard to inefficiency in input consumption or output production.

Increasing returns to scale (increasing RTS or IRS) play a basic role in competition and network economy.

Ramer (1986) claimed that profit growth alone may be the result of increased efficiency. Data envelopment analysis was first introduced by Charnes et al. (1978) [3]. After them, Banker et al. (1984) [1] introduced a model based on the variable returns to scale (VRS) assumption, and Färe (1957) proposed non-parametric models. Also, Färe (2000) discussed hyperbolic efficiency and its relationship with RTS. The concept of RTS in FDH models was introduced by Soleimani-damaneh and Mostafaee (2009) [10]. They investigated increasing and decreasing RTS in FDH models. Furthermore, Boussemart et al. (2009) [2] introduced homogeneous technologies and their relationship with α-returns to scale. In this paper, we propose the VRS model and α-returns to scale in homogenous technologies.

The paper consists of the following sections: In section 2, we review the DEA background. α -Returns to scale in the production technology with the VRS to scale assumption are presented in section 3. Finally, the conclusion is given in section 4.

2. DEA Background

2.1. Assumptions and definitions

Assume that we have n DMUs , DMU_j : j=1…n, to be evaluated, each DMU using m inputs to produce s outputs. $X_j = (x_{1j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, \dots, y_{sj})$ are the input and output vectors of $\mathrm{DMU_j}$, respectively, in which $X_j, Y_j \geq 0$, $X_j \neq 0$, and $Y_j \neq 0$.

The production possibility set T^t is represented as: $T^t = \{(X^t, Y^t) \in \mathbb{R}_+^{n+p} \mid \mathrm{x^t \text{ can produce} } y^t\}$

Let $L^t: \mathbb{R}_+^p \to 2^{\mathbb{R}_+^n}$ denote the input correspondence that maps all $y \in \mathbb{R}_+^p$ to input sets capable of producing them

$$L^t(y) = \{x \in \mathbb{R}_+^n : (x, y) \in T^t\}$$

Reciprocally, the output correspondence $P^t: \mathbb{R}_+^n \to 2^{\mathbb{R}_+^p}$ maps all $x \in \mathbb{R}_+^n$ into sets of outputs that can be produced by those inputs:

$$P^t(x) = \{y \in \mathbb{R}_+^n : (x, y) \in T^t\}$$

We have

$$(x, y) \in T^t \Leftrightarrow x \in L^t(y) \Leftrightarrow y \in P^t(x)$$

It is supposed that the production technology follows the axioms below:

T1: $(0,0) \in T^t, (0, y) \in T^t \Rightarrow y = 0$

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T2: The set $\mathrm{A}(x) = \{(u,v) \in T^t : u \le x\}$ of dominated observations is bounded.

T3: T^t is closed.

T4: For all $(x, y) \in T^t$ and all $(u,v) \in \Re_{+}^{n+p}$, we have $(-u, v) \le (-x, y) \Rightarrow (u, v) \in T^t$.

The following statement provides the definition of returns to scale (See Boussemart et al. (2009) [2]).

The production technology satisfies:

(1)

constant returns to scale if:

$$\forall \lambda \ge 0 \ ,(x^t, y^t) \in T^t \Rightarrow (\lambda x^t, \lambda y^t) \in T^t$$

(2)

non-decreasing returns to scale if:

$$\forall \lambda \ge 1 \ ,(x^t, y^t) \in T^t \Rightarrow (\lambda x^t, \lambda y^t) \in T^t$$

(3)

non-increasing returns to scale if:

$$\forall \lambda \in [0,1] \ ,(x^t, y^t) \in T^t \Rightarrow (\lambda x^t, \lambda y^t) \in T^t$$

In this paper, we employ definitions of strictly increasing/decreasing RTS that come from Boussemart et al. (2009) [2]. They used the definition of a weakly efficient frontier of the production possibility set proposed by Färe et al. (1985) [6],which is defined as follows:

$\omega(T^t) = \left\{ (x^t, y^t) \in T^t : (-x^t, y^t) \overset{*}{<} (-u^t, v^t) \Rightarrow (u^t, v^t) \notin T^t \right\}$. Notice that the symbol $\overset{*}{<}$ is defined as: $x \overset{*}{<} u \Leftrightarrow x \le u \quad and \ x_i < u_i, \ \forall i \in car(x) \ \cap I(u),$

$car(x) = \{i \in \{1,...,n\} : x_i > 0\}.$

So, they used the definitions of strictly increasing and strictly decreasing returns to scale. These are refinements of the definitions of increasing and decreasing returns to scale, respectively

2.2 Definition 1

Assume that in each time period t, the production possibility set T^t satisfies T_1-T_4. Then , T^t is said to exhibit:

(a)strictly increasing returns to scale if for all $(\lambda > 1)$

$(x^t, y^t) \in T^t, (x^t, y^t) \ne (0,0) \Rightarrow (\lambda x^t, \lambda y^t) \in T^t \setminus \omega(T^t)$

(b)strictly decreasing returns to scale if for all $(0 < \lambda < 1)$

$(x^t, y^t) \in T^t, (x^t, y^t) \ne (0,0) \Rightarrow (\lambda x^t, \lambda y^t) \in T^t \setminus \omega(T^t)$

This definition can be interpreted as follows: Boussemart et al considered the case where returns to scale are strictly increasing. Clearly Definition 1 (a) implies that returns to scale are increasing following condition (2). In addition definition1(a) imposes that the proportional expanding of an input-output vector is not sufficient to maintain it weakly efficient. A symmetrical analysis applies when returns to scale are strictly decreasing. First, Definition1(b) implies that returns to scale are decreasing according to condition (3). Moreover, a propotional contraction of an input-output vector is not sufficient to maintain it weakly efficient. This can connect returns to scale to a homogeneous technology. Now, we consider the definition of a homogeneous production technology proposed by Boussemart et al. (2009) [2].

A production technology T^t is said to be homogeneous of degree α if for all $(\lambda > 0)$: $(x^t, y^t) \in T^t \Rightarrow (\lambda x^t, \lambda^{\alpha} y^t) \in T^t$

Boussemart et al. showed that a direct connection can exist between a homogenous technology and RTS, so the following proposition can be proved.

2.2 Proposition1

Assume that in each time period t, the production possibility set T^t satisfies T_1-T_4. Moreover, suppose that T^t is homogeneous of degree α.

(a) If $\alpha > 1$, then T^t satisfies strictly increasing returns to scale.

(b) If $0 < \alpha < 1$, then T^t satisfies strictly decreasing returns to scale.

Proof

if the technology is homogeneous of degree α with $\alpha > 1$, then

$(x^t, y^t) \in T^t \setminus \{(0,0)\} \Rightarrow (\lambda x^t, \lambda^\alpha y^t) \in T^t$

For all $\lambda > 1$. But $\alpha > 1$ *and* $\lambda > 1$ implies that $\lambda^\alpha > \lambda$. Thus $(-\lambda x^t, \lambda y^t) \le (-\lambda x^t, \lambda^\alpha y^t)$.

Since the free disposal assumption holds, we deduce that $(\lambda x^t, \lambda y^t) \in T^t$. Moreover, since $(\lambda x^t, \lambda y^t) \notin \omega(T^t)$, we deduce that $(\lambda x^t, \lambda y^t) \in T^t \setminus \omega(T^t)$. We have that T^t satisfies strictly increasing returns to scale. (b) if $0 < \alpha < 1$, then $(x^t, y^t) \in T^t \setminus \{(0,0)\} \Rightarrow (\lambda x^t, \lambda^\alpha y^t) \in T^t$

For all positive $\lambda < 1$. However, $0 < \alpha < 1$ and $0 < \lambda < 1$ implies that $\lambda^\alpha > \lambda$. Thus, $(-\lambda x^t, \lambda y^t) \le (-\lambda x^t, \lambda^\alpha y^t)$. Since the free disposal assumption holds, we deduce that $(\lambda x^t, \lambda y^t) \in T^t$. Moreover, since $(\lambda x^t, \lambda y^t) \notin \omega(T^t)$, we have $(\lambda x^t, \lambda y^t) \in T^t \setminus \omega(T^t)$.

Consequently, we deduce that T^t satisfies strictly decreasing returns to scale .

Notice that the proof is similar to that of Boussemart et al. they said that if the technology is homogeneous of degree α then it satisfies α-returns to scale.

The concept of α-returns to scale, as defined by Boussemart et al. (2009) [2], is based upon an assumption of constant RTS. However, in this paper, we show in our corrected model that α-returns to scale can also be considered under the VRS assumption with suitable specification of λ.

3. **α-returns to scale in the production technology with variable returns to scale**

In this section, we show that the homogeneous production technology can be connected to α-returns to scale for limited λs when the production technology exhibits VRS.

3.1 Lemma 1

If DMU_o has IRS, then for all $\alpha > 1$ it has increasing α- returns to scale.

Proof

DMU_o, under evaluation, has IRS

So, we have:

$$(x, y) \in T, \forall \lambda > 1 \Rightarrow (\lambda x_o, \lambda y_o) \in T^t$$

Hence, the technology is homogeneous of degree α with $\alpha > 1$, then

$$\alpha > 1, \lambda > 1, \lambda^\alpha > \lambda \Rightarrow (\lambda x_o, \lambda^\alpha y_o) \in T^t$$

Moreover, since DMU_o has IRS and $(\lambda x_o, \lambda^\alpha y_o) \in T^t$, $(\lambda x_o, \lambda y_o) \in T^t \Rightarrow \lambda^\alpha y > \lambda y$

we will have

$(\lambda x_o, \lambda y_o) \in T^t \setminus \omega(T^t)$

Consequently, DMU_o exhibits increasing α- returns to scale.

Also, we can consider the case where decreasing RTS prevail.

Now we propose the following theorems for determining $\lambda \in (0, \lambda_o)$.

3.2 Theorem 1.

Consider the following two DMUs with VRS:

$$(\gamma_1 x, \gamma_1^\alpha y) \in T_v \quad , (\gamma_2 x, \gamma_2^\alpha y) \in T_v$$

Then we have

$$((\beta\, \gamma_1 + (1-\beta)\, \gamma_2)\, x, (\beta\, \gamma_1^\alpha + (1-\beta)\, \gamma_2^\alpha)\, y) \in T_v$$

Proof:

We know the following model T_v :

$$T_v = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x \ge \sum_{j=1}^{n} \lambda_j x_j \;\&\; y \le \sum_{j=1}^{n} \lambda_j y_j \;\&\; \sum_{j=1}^{n} \lambda_j = 1 \;\&\; \lambda \ge 0 \right\}$$

Corresponding to the above model and the convexity axiom:

$$\gamma_1 x \ge \sum_{j=1}^{n} \lambda_j x_j \;\&\; \gamma_1^\alpha y \le \sum_{j=1}^{n} \lambda_j y_j \;\&\; \sum_{j=1}^{n} \lambda_j = 1$$

$$\gamma_2 x \geq \sum_{j=1}^{n} \lambda_j x_j \quad \& \quad \gamma_2^{\alpha} y \leq \sum_{j=1}^{n} \lambda_j y_j \quad \& \quad \sum_{j=1}^{n} \lambda_j = 1$$

Then, we have

$$\beta\ \gamma_1\ x \geq \beta\ \sum_{j=1}^{n} \lambda_j x_j \quad \& \quad \beta\ \gamma_1^{\alpha} y \leq \beta\ \sum_{j=1}^{n} \lambda_j y_j \quad \& \quad \sum_{j=1}^{n} \lambda_j = 1$$

$$(1-\beta)\ \gamma_1\ x \geq (1-\beta)\ \sum_{j=1}^{n} \lambda_j x_j \quad \& \quad (1-\beta)\ \gamma_1^{\alpha}\ y \leq\ (1-\beta)\ \sum_{j=1}^{n} \lambda_j y_j \quad \& \quad \sum_{j=1}^{n} \lambda_j = 1$$

By the addition of the above relations, we get:

$$(\beta\ \gamma_1 + (1-\beta)\ \gamma_2)\ x \geq \sum_{j=1}^{n} (\beta\lambda_j + (1-\beta)\lambda_j)\ x_j$$

$$(\beta\ \gamma_1^{\alpha} + (1-\beta)\ \gamma_1^{\alpha})\ y \leq \sum_{j=1}^{n} (\beta\lambda_j + (1-\beta)\lambda_j)\ y_j$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

Therefore we conclude:

$$((\beta\ \gamma_1 + (1-\beta)\ \gamma_2)\ x, (\beta\ \gamma_2^{\alpha}, (1-\beta)\ \gamma_2^{\alpha})\ y) \in T_v\ .$$

We use the above theorem and show that the production technology in the BCC model is not compatible with α-returns to scale for each λ.

3.3 Lemma 2

For each > 0 :

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in T_v \qquad , \exists \lambda_0 > 0;\ \forall \lambda > 0\ , \lambda > \lambda_0 \Rightarrow (\lambda x, \lambda^{\alpha} y) \notin T_v$$

Proof:

We have $y_o \neq 0$. We assume there exists a k such that $y_{ko} \neq 0$, and we set $\wp = y_{kt} = \max\{y_{kj}\ , j = 1,...,n\}$

Regarding the above definition, we obviously have: $\forall (x, y) \in T_v \Rightarrow y_{kj} \leq y_{kt}$

We define $\lambda_0 = \left(\dfrac{\wp+1}{y_{ko}}\right)^{\frac{1}{\alpha}}$, therefore

$$(\lambda_0 x_o, \lambda_0{}^{\alpha} y_o) = \left(\lambda_0 x_o, \left((\frac{\wp+1}{y_{ko}})^{\frac{1}{\alpha}} \right)^{\alpha} y_o \right) = \left(\left(\frac{\wp+1}{y_{ko}}\right)^{\frac{1}{\alpha}} x, \left(\frac{\wp+1}{y_{ko}}\right) y \right) \notin T_v$$

because

$$\frac{\wp+1}{y_{ko}} \times y_{ko} = \wp + 1 > \wp$$

However, this contradicts:

$$\wp = y_{kt} = \max\{y_{kj}\ , j = 1,...,n\}.$$

This completes the proof.

3.4 Theorem 2

For each $\alpha > 0$:

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in T_v \qquad , \exists \lambda_0 > 0;\ \forall \lambda > 0\ , \lambda < \lambda_0 \Rightarrow (\lambda x, \lambda^{\alpha} y) \in T_v$$

Proof:

We consider set S as:

$$S=\{\bar{\lambda}>0\,|\,(\bar{\lambda}x,\bar{\lambda}^{\alpha}y)\in T_v\}$$

I) S is nonempty, because for $\bar{\lambda}=1$:

$(x,y)\in T_v$

II) S is convex (according to Theorem 3.2).

III) S is upper-bounded (according to Lemma 3.3).

Since S is nonempty and is an interval on $\mathbb{R}^n$, it is implied that S has a supremum. We assume $\lambda^*=\sup\ S$

(if $(x,y)\in\partial T_v\Rightarrow\lambda^*=1$)

Therefore, according to property of a supremum, for each $\lambda<\lambda^*$ we have:

$(\lambda x,\lambda^{\alpha}y)\in T_v$.

We consider set F as follows:

$$F=\{\alpha>0|(x_o,y_o)\,has\ increasing\ \alpha-returns\ to\ scale\}$$

3.5 *Theorem 3*

If DMU_o exhibits IRS, then it has increasing α_1-returns to scale and increasing α_2-returns to scale for $\alpha_1,\alpha_2\ (\alpha_1<\alpha_2)$; so for each $\alpha_1\le\alpha\le\alpha_2$, DMU_o has increasing α-returns to scale.

Proof:

DMU_o has α-IRS, for $\alpha>\alpha_1$, if it has α_1-IRS. This means: $\alpha_1>1\ ,\lambda>1\Rightarrow\ (\lambda x_o,\lambda^{\alpha_1}y_o)\in T_v$

Thus, $\alpha>\alpha_1\ ,\ \lambda^{\alpha}>\lambda^{\alpha_1}\Rightarrow\ (\lambda x_o,\lambda^{\alpha}y_o)\in T_v$

However, according to the definition of F, we have:

$\inf\ F=\alpha^*$,

So, for all $\alpha>\alpha^*$, RTS is increasing and for all $\alpha<\alpha^*$ RTS is decreasing.

If DMU_o has α_2-RTS, it means: $\alpha_2>1\ ,\ \lambda>1\Rightarrow(\lambda x_o,\lambda^{\alpha_2}y_o)\in T_v$

Therefore, $\alpha\le\alpha_2\Rightarrow\lambda^{\alpha}\le\lambda^{\alpha_2}\Rightarrow(\lambda x_o,\lambda^{\alpha}y_o)\in T_v$

Consequently

DMU_o has increasing α-returns to scale, which completes the proof.

According to the above discussion, in this paper we showed that the production technology with VRS is homogeneous for $\lambda\in(0,\lambda^*)$.

In the end, we propose the following lemma:

3.6. Lemma 3

(a) If $\alpha\to0$, then DMU_o satisfies decreasing α-returns to scale.

(b) If $\alpha\to\infty$, then DMU_o satisfies increasing α-returns to scale.

Proof:

According to the definition of α-returns to scale:

$$\lambda>1,\ (x_o,y_o)\in T^t\Rightarrow(\lambda x_o,\lambda^{\alpha}y_o)\in T^t$$

If $\lambda>1$, then we have:

$$\lim_{\alpha\to\infty}\ \lambda^{\alpha}=\infty$$

Considering the concept of IRS, we conclude that: $\lambda y_o<\lambda^{\alpha}y_o$. This means:

The α-returns to scale of DMU_o is increasing.

Now, if the following conditions is considered:

$$0<\lambda<1\ ,\ \lim_{\alpha\to0}\ \lambda^{\alpha}=1\ \Rightarrow\lambda^{\alpha}y_o<\lambda y_o$$

Then DMU_0 satisfies decreasing α-returns to scale.

4. Conclusion

The concepts of α-returns to scale and multi-output production technologies have been previously introduced in the literature (see Boussemart et al. (2009)). In this paper, we showed that the production technology with VRS satisfies α-returns to scale under a suitable specification of α and λ. Finally, in this context, we proposed some new theorems and proved them.

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