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# A New Approach to Compute Moments of Busy Period in $M^{[X]}/G/1/K$ Systems with Vacation Times

**M. Taremi*and M.R.Salehirad**

Faculty of Economics, Allameh Tabatabae'i University, Tehran, Iran

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**ABSTRACT**

One of the main characteristics in optimization and performance evaluation in a queuing model is its busy period. Finding the exact distribution of this variables in queuing models which service distribution and or inter-arrival distribution is general, is so complicated and there is usually no closed formula for it. Traditionally, in these situations, Laplase–Stieltjes transform this variable computed and then by using that the moments it determined. In this paper while, we address using this approach to computemoments of the duration of the busy period in $M^{[X]}/G/1/K$ systems with vacation times and under the partial batch acceptance strategy is associated with some important practical limitations, we propose a new procedure to calculate the moments of this queuing system. The new approach relies on the number of customers that arrive to the system during the first vacation time of system and taking full advantage of the Markov-regenerative process property. Numerical results are presented to demonstrate the simplicity and efficiency of the new approach.

**KEYWORDS**: $M^{[X]}/G/1/K$ Type queue, Laplase–Stieltjes transform, Moments, Vacation time, Acceptance strategy, Delay busy period.

## 1. INTRODUCTION

In the queuing models there are some features so that through them we can analyze the model. These characteristics are known as the effective sizes. On the other hand, optimization of queuing systems is done with the help of cost functions that are themselves functions of effective sizes. Also, since the arrival and service processes in queuing systems are random Therefore, effective measures are random. Hence, one of the important objectives in the analysis of queuing systems is obtained the probability distribution of this variable and or determined the characters of their distribution such as mean, variance and etc.

One of the important effective sizes in the queuing models, the busy period is. Finding the exact distribution of this variables in queuing models which service distribution and or inter-arrival distribution is general, is so complicated and there is usually no closed formula for that. Since that the moments determine the main feature of probability distribution for a random variable, so, this paper deals with the busy period moment in $M^{[X]}/G/1/K$ system with vacation times and under the partial batch acceptance strategy.

The batch-Poisson arrival $M^{[X]}/G/1/K$ finite capacity queue with server vacation is now common in telecommunications. For example, a processor (server) has secondary jobs (customers) to be performed aside from primary jobs. The processing time for a secondary job corresponds to a vacation time in queuing terminology. Another example is a buffer (queue) under the time division multiple access (TDMA) environment (see e.g., [13]). An arriving packet (customer) who finds the system idle cannot be transmitted (served) immediately, and it has to wait until the slotted boundary comes. A constant slotted time period corresponds to a vacation time. Performances issues in these examples then necessitate our queue with vacation time.

Traditionally, busy periods have been characterized through their Laplace-Stieltjes transforms [4], [6], [9], [11]. However, the Laplace-Stieltjes approach has some important practical limitations, that are not shared by procedure developed in this paper.

We analyze the duration of busy periods in $M^{[X]}/G/1/K$ systems by conditioning on the number of customers that arrive to the system during the first vacation time while, concurrently, taking full advantage of the

***Corresponding Author:** Mohammad Taremi, Faculty of Economics, Allameh Tabatabae'i University, Tehran, Iran.
Email address: sta.allameh.uni@gmail.com

Markov-regenerative structure of the number of customers in $M^{[X]}/G/1/K$ systems (see, e.g., [7] for the definition and properties of Markov regenerative processes).

By busy period it is (usually) meant the period of time that starts when a customer arrives to an empty system and ends at the first subsequent time at which the system becomes empty (see, e.g., [1]. [12]). In this paper, we obtain results for the slightly generalized case of delay busy periods (see, e.g., [3]), where a busy period is initiated by some task other than the processing of an ordinary job.

We end this introduction with a brief outline of the paper. In Section II we introduce the studied our model, along with some relevant notation. In Section III we motivate and state the main result of the paper, which can be used to compute moments of the duration of delay busy periods of $M^{[X]}/G/1/K$ systems, as it is illustrated through example in Section IV.

## 2. The MODEL AND SOME NOTATION

We consider an $M^{[X]}/G/1/K$ queue, where $K$ equals the number of waiting places in the queue, including the space for the customer that may be in service. We assume that the arrival epochs of the batches form a homogeneous Poisson process with intensity $\lambda$ and the service times form a sequence of i.i.d. random variables with general distribution function $G(.)$. Service times are independent of the arrival process and are not affected by the discipline, and Customers accepted by the system are served by a single server exhaustively, i.e. the server serves the queue continuously until the queue is empty. As regards the customer acceptance policy, we consider what is known as partial blocking ( see, e.g.,[14]) in which if at arrival of a batch of $l$ customers there are only $m, m<l$, free position available in the system, then $m$ customers of the batch enter the system and the remaining $l-m$ customers of the batch are blocked.

In the case of delay busy periods, at a time when there are no customers in the system, the server is made unavailable for a time $V$, having distribution function $V(.)$. At the end of the delay (vacation) the server begins work with a backlog of however many customers (possibly none) have arrived during the delay and continues until there are none left.

We let $(f_i)_{i\in N_+}$ denote the batch size probability function, where $N_+=\{1,2,3,...\}$, and $f_i^{(r)}$ denotes the probability that the total number of customers in *r* customer batches is equal to *i*. Note that $f_a^{(0)}=\delta_{0a}$, and

$$f_a^{(r)}=\sum_{i=r-1}^{a-1} f_{a-i}\, f_i^{(r-1)}$$

for $r\in N_+$ and $a=r,r+1,...,$ where $\delta_{ia}$ is the Kronecker delta function, i.e., $\delta_{ia}=1$ if $i=a$ and $\delta_{ia}=0$ otherwise.

In addition, we let $p_a, a\in N=\{0,1,2,...\}$, denote the probability that *a* customers arrive during the vacation time. Then, by conditioning on the number of batches arriving during the vacation time, we have

$$p_a=\sum_{r=0}^{a} f_a^{(r)}\alpha_r \tag{1}$$

where $\alpha_r$ is *r*–th mixed-Poisson probability with arrival rate $\lambda$ and mixing distribution $G(.)$, i.e. (see, e.g., [5],[8]),

$$\alpha_r(S)=\int_0^\infty \frac{e^{-\lambda s}(\lambda s)^r}{r!}\,dG_S(s). \tag{2}$$

## 3. MOMENTS OF THE DURATION OF A BUSY PERIOD

Let $B_{i,K}$ be the duration of a busy period that starts with a backlog of *i* jobs when the queue has a total capacity of *K* jobs waiting. In this notation, $B_{1,K}$ corresponds to the ordinary busy period. $B_{i,K}$ is independent of the discipline so that the busy period is equivalent to the sum of $i$ busy periods, the $j$-th one having length $B_{1,K-i+j}$ because $i-j$ spaces are occupied by original jobs. Thus,

$$B_{i,K} = \bigoplus_{j=\text{K+1-i}}^{K} B_{1,\text{j}} \tag{3}$$

where $\bigoplus$ denotes convolution of random variables, i.e., the sum of independent random variables.

Now, considering the random variable representing the duration of the delay busy period with $K$ waiting spaces by $D_K$, its Laplase–Stieltjes transform by $\delta_K(t)$ and distribution function and Laplase–Stieltjes transform of $B_{1,K}$ respectively by $B_{1,j}(.)$ and $\beta_{1,j}(t)$, we derive the following result.

**Theorem 1.** The Laplase–Stieltjes transform for the distribution of the duration of a delay busy period for an $M^{[X]}/G/1/K$ queue is

$$\delta_K(t) = v_0(t) + \sum_{a=1}^{K-1} \omega_a(t) \prod_{j=K-a+1}^{K} \beta_{1,j}(t) + \sum_{a \ge K} \omega_a(t) \prod_{j=1}^{K} \beta_{1,j}(t), \tag{4}$$

where $\omega_a(t) = \int_0^\infty \sum_{r=0}^{a} f_a^{(r)} \frac{(\lambda v)^r}{r!} e^{-(t+\lambda)v} dV(v)$.

**Proof.** Let $A$ be the number of arrivals during the vacation time. Then

$$E\left[\exp(-tD_K)\middle| V = v, A = a, B_{\min(a,K),K} = x\right] = \exp[-t(v+x)],$$

where $x=0$ if $a=0$. For $1 \le a \le K$, $B_{a,K}$ can be replaced by its decomposition, given by (3). Then integrate with respect to $B_{1,j}(.)$, $K+1-a \le j \le K$, and apply the convolution theorem for Laplace transform so that $\exp(-tx)$ is replaced with products of the $\beta_{1,j}(t)$. Finally, weight each term by the probability associated with the number of arrivals during the vacation time and integrate with respect to $V(.)$.

■

However, moments of our model may be obtained by differentiation of (4), but, this procedure for computing the higher moments, especially, when increasing capacity of the system, the complicated computation is needed. In the following theorem, we computation this moments by another way.

**Theorem 2.** The integer moments of the duration of delay busy periods for an $M^{[X]}/G/1/K$ queue is

$$\begin{aligned} E(D_K^m) = E(V^m) &+ \sum_{a=1}^{K-1} p_a E(B_{a,K}^m) + \sum_{a=1}^{K-1} p_a \sum_{j=1}^{m-1} \binom{m}{j} E(\overline{V}_a^{\,j}) E(B_{a,K}^{m-j}) \\ &+ \sum_{a \ge K} p_a E(B_{K,K}^m) + \sum_{a \ge K} p_a \sum_{j=1}^{m-1} \binom{m}{j} E(\overline{V}_a^{\,j}) E(B_{K,K}^{m-j}), \end{aligned} \tag{6}$$

where $m \in N$.

**Proof.** We let $\overline{V}_a$ denote the duration of the vacation time of the first delay busy period given that exactly $a$ customers arrive to the system during his delay. If no customers arrive to the system during the vacation time of the first delay busy period, the delay busy period ends with terminate the vacation time. Otherwise, the customers that arrive to the system during the vacation time and are not blocked initiate, at end the vacation time, a multiple-busy period that is part of the first delay busy period under consideration and adds to the duration of the vacation time. Namely,

$$(D_K \mid A = a) \overset{d}{=} \overline{V}_c \oplus B_{\min(a,K),K},$$

where $\overset{d}{=}$ denotes equality in distribution.

Taking into account that has probability function $p_a$, (1) leads to

$$E(D_K^m) = \sum_{a=0}^{\infty} p_a \, E(\overline{V}_a \oplus B_{\min(a,K),K})^m$$

with $B_{0,K}$ denoting a random variable that is null with probability one. By using Newton's binomial formula in previous equation, we have

$$E(D_K^m) = p_0 E(\overline{V}_0^m) + \sum_{a=1}^{K-1} p_a \sum_{j=0}^{m} \binom{m}{j} E(\overline{V}_a^j) E(B_{a,K}^{m-j})$$
$$+ \sum_{a \geq K} p_a \sum_{j=0}^{m} \binom{m}{j} E(\overline{V}_a^j) E(B_{K,K}^{m-j})$$

By separating the terms for which $j=0$ and $j=m$ from the remaining terms in the previous equation, taking into account $E(V^m) = \sum_{a \geq 0} p_a E(\overline{V}_a^m)$, we conclude equation (6).

■

As it mentioned, Pacheco and Riberio [10] has addressed arecursive algorithm to compute moments $E(B_{a,K}^m), 1 \leq a \leq K.$ In following theorem, we provide a simpler form for the computation the coefficients $\left( p_a E[\overline{V}_a^l] \right)_{0 \leq a \leq K-2, 0 \leq l \leq m}$.

**Theorem 3.** The absolute moment of order *m*, $m \in N_+$, of conditional random variable $\overline{V}_a$, verifies

$$p_a E[\overline{V}_a^m] = \sum_{j=0}^{a} \frac{(m+j)!}{\lambda^m j!} \alpha_{m+j} f_a^{(j)}, \tag{7}$$

for $a \in N$, and, moreover,

$$\sum_{a \geq K} p_a E(\overline{V}_a^m) = E(V^m) - \sum_{a=0}^{K-1} \sum_{j=0}^{a} \frac{(m+j)!}{\lambda^m j!} \alpha_{m+j} f_a^{(j)}. \tag{8}$$

**Proof.** For $m \in N_+$ and $a \in N$,

$$\begin{aligned} p_a E[\overline{V}_a^m] &= E\left[ V^m 1_{\{A=a\}} \right] \\ &= \int_0^{\infty} u^m \sum_{j=0}^{a} e^{-\lambda u} \frac{(\lambda u)^j}{j!} f_a^{(j)} dV(u) \\ &= \sum_{j=0}^{a} \frac{(m+j)!}{\lambda^m j!} \int_0^{\infty} e^{-\lambda u} \frac{(\lambda u)^{m+j}}{(m+j)!} dV(u) \, f_a^{(j)} \\ &= \sum_{j=0}^{a} \frac{(m+j)!}{\lambda^m j!} \alpha_{m+j} f_a^{(j)} \end{aligned}$$

Finally, equation (8) follows from equation (7) since $E(V^m) = \sum_{a \geq 0} p_a E(\overline{V}_a^m)$, thus implying that

$$\sum_{a \geq K} p_a E(\overline{V}_a^m) = E(V^m) - \sum_{a=0}^{K-1} p_a E(\overline{V}_a^m).$$

■

The most immediate application of Theorem 2 is for the computation of the expected value of the duration of the delay busy period of $M/G/1/K$ systems, in which case we conclude that

$$E(V_K) = E(V) + \sum_{j=1}^{K} E\left(B_{1,j}\right) \sum_{a=K-j+1}^{\infty} \alpha_a.$$

This leads to equation derived by Miller [9] to compute the duration of a delay busy period of an $M/G/1/K$ system.

## 4. UMERICAL ILLUSTRATION

In this section, we use the results proposed in the previous sections to compute moments of the duration of delay busy periods in $M^{[X]}/G/1/K$ systems and illustrate the sensitivity of their associated duration of delay busy periods respect to service time distributions and traffic intensity.

Specifically, to evaluate the influence of the service time distributions we consider the following the service time distributions with common mean with positive mean $\mu^{-1}$: deterministic with value $\mu^{-1}$, $D(\mu^{-1})$; exponential with rate $\mu$, $M(\mu)$; Erlang with $k$ phases, $E_k = E_k(k\mu)$; and, Pareto with parameters $(\beta, k)$ with $\beta > 1$ and $k = (\beta - 1)/\beta\mu$, $P(\beta, (\beta-1)/\beta\mu)$. We let also $EV_K$ $\left(CV_K, SV_K, \text{ and } KV_K\right)$ denote the expected value or mean (coefficient of variation, skewness and kurtosis) of the duration of delay busy period (see,e.g.,[2]).
Table **I** is relative to single arrival $M/G/1/K$ systems withbatch arrival rate $\lambda = 0.95$ ,unit service rate and exponential vacation time distribution with unit mean. This table shows how the duration of 1-busy periods evolves as the systemcapacity increases.

**Table I**. Expected value, coefficient of variation, skewness and kurtosis of the duration of delay busy period in $M/G/1/K$ systems with arrival rate $\lambda = 0.95$, unit Service rate and exponential vacation time distribution with unit mean

| | $M/M(1)/1/K$ | | | | $M/D(1)/1/K$ | | | |
|---|---|---|---|---|---|---|---|---|
| $K$ | $EV_K$ | $CV_K$ | $SV_K$ | $KV_K$ | $EV_K$ | $CV_K$ | $SV_K$ | $KV_K$ |
| 1 | 1 | 1 | 2 | 9 | 1 | 0 | - | - |
| 2 | 1.9500 | 1.2246 | 2.3238 | 10.7353 | 2.5857 | 0.7831 | 2.0601 | 9.2439 |
| 3 | 2.8525 | 1.4525 | 2.7064 | 13.2209 | 4.2295 | 1.1130 | 2.3983 | 11.1232 |
| 4 | 3.7099 | 1.6573 | 3.0727 | 16.0158 | 5.7514 | 1.3688 | 2.7922 | 13.7982 |
| 5 | 4.5244 | 1.8420 | 3.4149 | 18.9797 | 7.1296 | 1.5846 | 3.1724 | 16.8247 |
| 6 | 5.2982 | 2.0108 | 3.7354 | 22.0580 | 8.3744 | 1.7736 | 3.5309 | 20.0654 |
| 11 | 8.6240 | 2.6995 | 5.1128 | 38.5952 | 13.0093 | 2.4914 | 5.0778 | 38.3280 |
| 16 | 11.1975 | 3.2334 | 6.2691 | 56.6381 | 15.7963 | 2.9953 | 6.3795 | 59.2408 |
| 21 | 13.1888 | 3.6739 | 7.3009 | 76.0173 | 17.4722 | 3.3691 | 7.5286 | 82.2161 |
| 26 | 14.7296 | 4.0475 | 8.2491 | 96.6351 | 18.4800 | 3.6485 | 8.5520 | 106.5821 |
| 31 | 15.9219 | 4.3684 | 9.1339 | 118.3832 | 19.0860 | 3.8555 | 9.4559 | 131.5239 |
| 36 | 16.8444 | 4.6456 | 9.9655 | 141.1300 | 19.4504 | 4.0068 | 10.2410 | 156.1573 |
| 41 | 17.5583 | 4.8856 | 10.7498 | 164.7177 | 19.6695 | 4.1158 | 10.9088 | 179.6343 |
| 46 | 18.1106 | 5.0931 | 11.4895 | 188.9643 | 19.8013 | 4.1929 | 11.4637 | 201.2428 |
| 51 | 18.5380 | 5.2721 | 12.1859 | 213.6676 | 19.8805 | 4.2468 | 11.9141 | 220.4774 |
| $\infty$ | 20 | 6.2450 | 18.7309 | 587.7692 | 20 | 4.3589 | 13.3061 | 298.0526 |

For illustration of the effects of service time distributionsand traffic intensityin the duration of delay busy periods, we consider in Figure**I** $M^{D(3)}/G/1/20$ systemsfor several service time distributions: deterministic, geometric, shifted binomial anddiscrete uniform distributions. The systems considered have deterministic batch size distribution with mean 3, batch arrival rate 1/3 and exponential vacation time distribution with unit mean

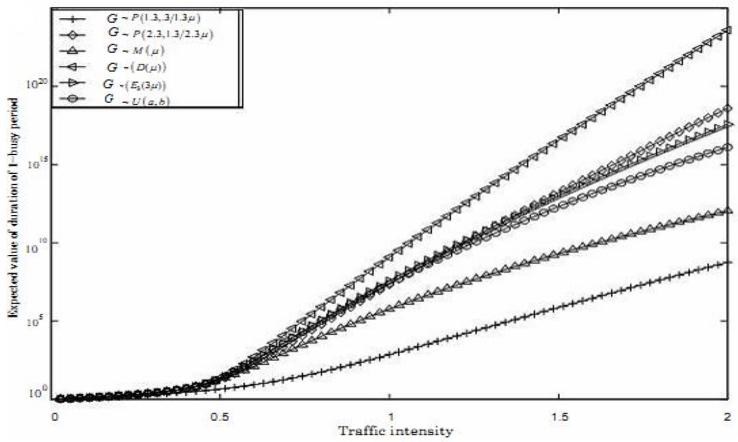


**Figure I**: Expected value of durations of busy periods of $M^{D(3)}/G/1/20$ systemswith mean batch size 3, batch arrival rate 1 /3 and exponential vacation time distribution with unit mean.

## 5. CONCLUSION

In this study, we addressed batch arrival $M^{[X]}/G/1/K$ systems with vacation times and showed that using the Laplase–Stieltjes transform method to calculate the moments of the duration of the busy period in this system is associated with some limitations. Therefore, we have proposed a new approach to calculate the moments of the system. The newapproachrelieson the number of customers that arrive to the system during the first vacation time of system and taking full advantage of the Markov-regenerative process property. This approach also provided a systematic recursive function to calculate the moments of the system.

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