Free Vibration Analysis of Cable Using Adomian Decomposition Method

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ABSTRACT

Adomian decomposition method is employed to investigate the free vibration of cable consisting of two uniform sections. Each section is considered a substructure which can be modeled using ADM. Applying boundary conditions and continuity condition, the dimensionless natural frequencies and corresponding mode shapes may be obtained simultaneously. Computed results for different boundary conditions are presented which agree well with those analytical ones given in the literature. Also the effect of amount of mass and spring stiffness at the boundary on variation of natural frequencies is obtained and an effective procedure is suggested for encountering with these kinds of problems.

KEYWORDS: Cable; Adomian Decomposition Method; Vibration; Natural frequency; Mode shape.

INTRODUCTION

Nonlinear phenomena play an important role in applied science. Many of the analytical solution methods to solve nonlinear differential equations use linearization method or neglect the nonlinearity effects and assume they are relatively insignificant; however, this assumption can change the real solution of the mathematical model which describes the physical phenomena very seriously. Also numerical methods are based on discretization technique which is not effective in all states of time and space variable ranges. A large class of analytical solution methods and numerical solution methods have been used to solve these problems such as Hirota’s bilinear method\cite{1}, The symmetry method\cite{2}, the Darboun transformation\cite{3}, the Homotopy perturbation method\cite{4-6}, variational iteration method\cite{7}, Adomian decomposition method\cite{8, 9} and other numerical methods.

In this study, a new computed approach called Adomian decomposition method (ADM) is introduced to solve the vibration problem of uniform cables. ADM have been proved to be effective and reliable for solving linear or nonlinear equations. \cite{8, 10-16}. Recently, the ADM has been applied to the problem of vibration of mechanical and structural systems \cite{17-21}.

Unlike the perturbation method, ADM does not depend on any parameter and gets the solution as an infinite series which converges to the exact solution \cite{22}. The chief advantage of ADM is that it does not need any linearization, discretization and any other restrictive assumptions; consequently the reality of the problem does not change due to linearization, also the solution is not affected by errors related to descretization. Another advantage of ADM is its rapid convergence. That it only needs a small number of terms of series to get an approximation of the solution with high accuracy \cite{23, 24, 25}.

In this article we have used this method to solve the transverse vibration of uniform cables which governs numerous engineering experiments.

Using ADM, the two coupled governing differential equations become two recursive algebraic equations and the boundary conditions at the right and the left ends; become two simple algebraic frequency equations which are suitable for symbolic computation. Moreover, after some simple algebraic operations on these frequency equations, any \textit{i}th natural frequency and the corresponding mode shape may be obtained one at the same time. Finally some problems of uniform cables are solved both analytically and by using ADM. Results obtained from ADM, show excellent agreement with analytical ones which verify the accuracy and efficiency of the method.

ADM has been the focus of many researches in the last decades. Ismail et al\cite{26} used ADM to solve Burger's-Huxley and Burger's-Fisher equations. The Shakeri\cite{27} and Dehghan have employed ADM to solve a system of two nonlinear integral-differential equations which has special importants in biology. These equations describe a model of biological species living.

Adomian and Ratch \cite{28} introduced the phenomena of the so called noise terms. They concluded that noise terms always appear in inhomogeneous equations and if a term or terms in the part \(y_0\) are cancelled by a term or terms in the part \(y_1\), then the remaining terms in \(y_0\) can get the exact solution \(y(x)\). This conclusion may be useful in reaching a fast convergence of the series of solutions.

Hosseini\cite{29} and Nasabzadeh, have introduced an efficient modification of Adomian decomposition method for solving second order ordinary differential equations. They showed that the proposed method can be applied to singular and nonsingular problems.

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Wazwaz [13] has introduced the modified Adomian decomposition method to solve some kinds of nonlinear problems.

Nuguleswaran [30], obtained an approximate solution for the transverse vibration of a uniform Euler-Bernoulli beam under linearly varying axial force. The converges of method is discussee by Tatari and Dehghan [31].

2. METHODS

2.1. Adomian Decomposition Method

In this section, the ADM is briefly explained. Consider a general differential equation:

\[ Ly + Ry = g(x) \]  

(1)

Where \( L \) is an invertible operator which contains highest order of derivatives and \( R \) contains reminder orders of derivatives and \( g(x) \) is the specific function. Solving for \( Ly \), one can obtain:

\[ y = \Phi + L^{-1}g - L^{-1}Ry \]  

(2)

In Eq. (2), \( \Phi \) is integration constant such that \( 0 = \Phi L \). For solving Eq. (2) by ADM, \( y \) can be written as series \( \sum_{k=0}^{\infty} y_k \). Substituting this series into Eq. (2) yields:

\[ \sum_{k=0}^{\infty} y_k = \Phi + L^{-1}g - L^{-1}R \sum_{k=0}^{\infty} y_k \]  

(3)

In the above equation, by assuming \( y_0 = \Phi + L^{-1}g \), the recursive formula is obtained as follows:

\[ y_k = -L^{-1}Ry_{k-1}, \quad k \geq 1 \]  

(4)

2.2. Cable vibration theorem

Consider a pre-tensioned uniform cable with length \( l \). For this cable which has the constant tension \( P \) and is subjected to the lateral force \( f(x,t) \), the governing equation of motion is as follows:

\[ \rho \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^2 w(x,t)}{\partial x^2} + f(x,t) = 0 \]  

(5)

Where \( w(x,t) \) and \( \rho \), are transverse displacement and density of the cable respectively, and \( t \) shows the time. For \( f(x,t) = 0 \), free vibration equation may be written as:

\[ \rho \frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} \]  

(6)

Or

\[ c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \]  

(7)

In which:

\[ c = \left( \frac{P}{\rho} \right)^{1/2} \]  

(8)

Using separation of variables technique, Eq. (7) may be written as Eqs. (9) and (10).

\[ \frac{d^2 y(x)}{dx^2} + \omega^2 y(x) = 0 \]  

(9)

\[ c^2 \frac{dT(t)}{dt} + \omega^2 T(t) = 0 \]  

(10)

In which \( \omega \) is the frequency of vibration. Eqs. (11) and (12) are deduced by solving Eqs. (9) and (10):

\[ y(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \]  

(11)

\[ T(t) = C \cos \omega t + D \sin \omega t \]  

(12)

Constants A, B, C and D will be obtained from boundary/initial conditions.

3. Formulation of cable vibration by ADM method:

The cable shown in Fig.1 is considered. The cable is divided in two sections with the same length and the midpoint of the cable is considered as origin of the coordinate system.

![Figure 1. A two sectioned cable](image)
Dimensionless equation of motion for each part of the cable can be written as:
\[
\frac{d^2y_i(x_i)}{dx_i^2} - \lambda y_i(x_i) = 0 \quad , \quad i = 1,2
\]  
(13)

In which:
\[
\lambda = \frac{\alpha^2 \rho l_i^4}{E A} \quad , \quad X_i = \frac{x_i}{l_i} \quad , \quad Y_i = \frac{y_i}{l_i} \quad , \quad i = 1,2
\]  
(14)

Note that i refers to the section considered.

Operators L and R are defined as:
\[
L_i y_i(x_i) = \frac{d^2y_i(x_i)}{dx_i^2} \quad , \quad R_i y_i(x_i) = -\lambda y_i(x_i) \quad , \quad i = 1,2
\]  
(15)

Applying ADM to the Eq. (13) we get:
\[
\sum_{i=1}^{2} Y_i(x_i) = \phi + \lambda L_i \sum_{i=1}^{2} Y_i(x_i) \quad , \quad i = 1,2
\]  
(16)

In which:
\[
L_i^{-1} = \iiint dX_i \quad , \quad i = 1,2
\]  
(17)

As mentioned before, the first term of the left side series is considered equal to \( \phi \):
\[
Y_i(x_i) = \phi = y_i(0) + Y_i^{(0)}(0)X_i \quad , \quad i = 1,2
\]  
(18)

Therefore, the recursive relation is obtained as:
\[
Y_i(x_i) = \lambda L_i^{-1} Y_{i+1} (x_i) = \lambda \sum_{\alpha=1}^{\infty} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha! \sum_{\alpha=1}^{N} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha!}} Y_i^{(\alpha+\alpha)}(0) \quad , \quad i = 1,2
\]  
(19)

Substituting Eq. (18) as the first term into Eq. (19) and expanding other terms, \( Y_i(x_i) \) is obtained as follows:
\[
Y_i(x_i) = \lambda L_i^{-1} Y_{i+1} (x_i) = \lambda \sum_{\alpha=1}^{\infty} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha! \sum_{\alpha=1}^{N} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha!}} Y_i^{(\alpha+\alpha)}(0) \quad , \quad i = 1,2
\]  
(20)

After achieving the general relations of each equations, \( Y_i(x_i) \) can be approximated as follows:
\[
Y_i(x_i) = \sum_{\alpha=1}^{\infty} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha! \sum_{\alpha=1}^{N} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha!}} Y_i^{(\alpha+\alpha)}(0) \quad , \quad i = 1,2
\]  
(21)

Continuity conditions at midpoint are expressed as:
\[
y_i(0) = y_i(0) \quad , \quad y_i^{(0)}(0) = y_i^{(0)}(0)
\]  
(22)

According to Eq. (14), dimensionless Eq. (22) would be:
\[
Y_i(0) = \frac{1}{\alpha} Y_i(0) \quad , \quad Y_i^{(0)}(0) = Y_i^{(0)}(0)
\]  
(23)

In which:
\[
\alpha = \frac{l_i}{l_i}
\]  
(24)

Substituting Eq. (23) into Eq. (21) yields:
\[
Y_i(x_i) = \sum_{\alpha=1}^{\infty} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha! \sum_{\alpha=1}^{N} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha!}} Y_i^{(\alpha+\alpha)}(0)
\]  
(25)

\[
Y_i(x_i) = \sum_{\alpha=1}^{\infty} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha! \sum_{\alpha=1}^{N} \frac{X_i^{(\alpha+\alpha) - (\alpha+\alpha)!}}{\alpha!}} Y_i^{(\alpha+\alpha)}(0)
\]  
(26)

Applying boundary conditions at both ends, a homogenous system of equations with two unknowns is got. In order to have a nontrivial solution for the system of equations, the determinant of the matrix of coefficients must be zero. The natural frequencies of system can be obtained by solving the resultant equation. By substituting any \( n \)th natural frequency into the aforementioned homogenous system of equations, the corresponding eigen vectors is calculated. The \( n \)th mode shape is also gained by substituting these vectors into Eqs. (25) and (26).

4. Examples
4.1. Free vibration of a cable with both ends pinned
4.1.1. Analytical solution

According to Fig.2, boundary conditions are:
\[
w(0,t) = w(l,t) = 0
\]  
(27)

Or
Frequency equation is obtained by applying Eq. (28) to Eq. (11):

\[
\omega_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots.
\]

(29)

4.1.2. Solution using ADM

As it is shown in Fig. 3, dimensionless boundary conditions are:

\[
Y_1(1) = 0, \quad Y_1(-1) = 0
\]

(30)

Applying Eq. (30) to Eqs. (25) and (26), matrix form of governing equations will be got as follows:

\[
\begin{bmatrix}
\sum_{k=1}^{\infty} \frac{(-\eta_1)^k \omega^{2k}}{k!} \\
\sum_{k=1}^{\infty} \frac{(-\eta_1)^k \omega^{3k}}{k!}
\end{bmatrix}
\begin{bmatrix}
Y_1(0) \\
Y_1'(0)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(31)

In which:

\[
\lambda_i = \frac{l \rho \omega^2}{P}, \quad i = 1, 2
\]

(32)

Assuming \(P = 4000N, \rho = \rho_1 = 5Kg/m, l_1 = l_2 = 1m\), natural frequencies can be calculated. They are presented in table 1 along with the exact solutions. The value of \(n\) in the table is chosen arbitrary. According to table 1, it is found that the number of obtained frequencies and their accuracy is increased and converged to the exact value by increasing the value of \(n\). The three first mode shapes are shown in Fig. 4.

Figure 2. The uniform cable with both ends pinned used in analytical solution

Figure 3. The cable with both ends pinned used in ADM solution

Figure 4. Three first mode shapes of the cable with both ends pinned
Table 1. Natural frequencies of the cable with both ends pinned

<table>
<thead>
<tr>
<th>n</th>
<th>( \omega_1 ) (rad/s)</th>
<th>( \omega_2 ) (rad/s)</th>
<th>( \omega_3 ) (rad/s)</th>
<th>( \omega_4 ) (rad/s)</th>
<th>( \omega_5 ) (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>45.0413</td>
<td>87.0131</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>44.4288</td>
<td>88.8576</td>
<td>132.5547</td>
<td>169.0933</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7153</td>
<td>222.1441</td>
</tr>
<tr>
<td>18</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7153</td>
<td>222.1441</td>
</tr>
</tbody>
</table>

4.2. Free vibration of a cable with one end pinned - one end free

4.2.1. Analytical solution

According to Fig. 5, one end at \( x=0 \) is fixed and its displacement is set to zero. Another end is pinned to a joint which can freely move along vertical direction. This end cannot sustain lateral force. Thus the boundary conditions would be:

\[
\frac{\partial w(l,t)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \theta(l,t)}{\partial x} = 0
\]

Or

\[
y(0) = \frac{\partial y(l)}{\partial x} = 0
\]

Figure 5. The uniform cable with one end pinned - one end free used in analytical solution

Applying Eq. (34) to Eq. (11), we get frequency equation as:

\[
\omega_n = \frac{(2n+1)c\pi}{2l}, \quad n = 0,1,2,....
\]

(35)

4.2.2. Solution using ADM

Dimensionless boundary conditions as shown in Fig. 6 are:

\[
Y_1^{(i)}(1) = 0 \quad \text{and} \quad Y_i(-1) = 0
\]

(36)

Figure 6. The cable with one end pinned - one end free used in ADM solution

\( Y_1^{(i)} \) can be calculated by differentiating from Eq. (25). Matrix form of governing equations can be presented as Eq. (37) by applying Eq. (36) to Eqs. (25) and (26).

\[
\begin{bmatrix}
\sum_{r=0}^{\infty} \left(-\eta_r\right)^r \frac{\alpha^{2r}}{(2k+1)!} \\
\sum_{r=0}^{\infty} \left(-\eta_r\right)^r \frac{\omega^{2r}}{(2k+1)!} \\
\sum_{r=0}^{\infty} \left(-\eta_r\right)^r \frac{\omega^{2r}}{(2k+1)!} \\
\sum_{r=0}^{\infty} \left(-\eta_r\right)^r \frac{\omega^{2r}}{(2k+1)!}
\end{bmatrix}
\begin{bmatrix}
Y_1(0) \\
Y_1^{(i)}(0)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(37)

Imagining scalar parameters of Ex. 1, natural frequencies and related mode shapes (three first mode shapes) are gained and shown in Table 2 and Fig. 7, respectively.
4.3. Free vibration of a cable with one end pinned- one end sprung

4.3.1. Analytical solution

The cable is pinned at the left end and is connected to a spring with stiffness constant $k$ at the other end as shown in Fig. 8. Boundary conditions would be:

$$w(0,t) = 0, \quad P\frac{dw(l,t)}{dx} = -kw(l,t) \quad (38)$$

Or

$$y(0) = 0, \quad P\frac{dy(l)}{dx} = -ky(l) \quad (39)$$

![Figure 8. The uniform cable with one end pinned- one end sprung used in analytical solution](image)

Natural frequencies can be obtained from solving the following frequency equation:

$$\tan \frac{\omega l}{c} + \frac{P\omega}{Kc} = 0 \quad (40)$$

In which $P$ is the constant tension of cable, $K$ is stiffness constant and $c$ is equal to $(P/\rho)^{1/2}$. It is evident that Eq. (40) is nonlinear which does not have an explicit solution and numerical techniques should be applied for solving it.

4.3.2. Solution using ADM

According to Fig. 9, boundary conditions are considered as follows:

$$P_{Y(l)}^\text{free} = -kY(l), \quad Y(1) = 0 \quad (41)$$

Table 2. Natural frequencies of the cable with one end pinned- one end free

<table>
<thead>
<tr>
<th>n</th>
<th>$\omega_1$ (rad/s)</th>
<th>$\omega_2$ (rad/s)</th>
<th>$\omega_3$ (rad/s)</th>
<th>$\omega_4$ (rad/s)</th>
<th>$\omega_5$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>21.2973</td>
<td>58.1929</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>22.2452</td>
<td>64.8998</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>22.2144</td>
<td>66.6431</td>
<td>110.9562</td>
<td>154.7517</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>22.2144</td>
<td>66.6432</td>
<td>111.0716</td>
<td>155.5296</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>22.2144</td>
<td>66.6432</td>
<td>111.0720</td>
<td>155.5009</td>
<td>199.9297</td>
</tr>
<tr>
<td>exact</td>
<td>22.2144</td>
<td>66.6432</td>
<td>111.0720</td>
<td>155.5009</td>
<td>199.9297</td>
</tr>
</tbody>
</table>

The cable is pinned at the left end and is connected to a spring with stiffness constant $k$ at the other end as shown in Fig. 8. Boundary conditions would be:

$$w(0,t) = 0, \quad P\frac{dw(l,t)}{dx} = -kw(l,t) \quad (38)$$

Or

$$y(0) = 0, \quad P\frac{dy(l)}{dx} = -ky(l) \quad (39)$$

![Figure 8. The uniform cable with one end pinned- one end sprung used in analytical solution](image)
Substituting Eq. (41) into Eqs. (25) and (26), we would have matrix form of governing equations:

\[
\begin{bmatrix}
\sum (-\eta)^{\nu} \omega^{\nu} \mu^{(2k-1)\nu!} + \sum (-\eta)^{\nu} \omega^{\nu} \mu^{(2k)\nu!} + \sum (-\eta)^{\nu} \omega^{\nu} \mu^{(2k+1)\nu!} \\
\sum (-\eta)^{\nu} \omega^{\nu} \mu^{(2k)\nu!} + \sum (-\eta)^{\nu} \omega^{\nu} \mu^{(2k+1)\nu!} \\
\end{bmatrix}
\begin{bmatrix}
Y_1(0) \\
Y_2(0) \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

Considering scalar parameters of Ex. 1 and \( K_i = K = 16000 \ell^2/m \), natural frequencies and the three first mode shapes are obtained. The results are shown in table 3 and Fig. 10 respectively.

Obtained results are validated by satisfying the natural frequencies derived from ADM in Eq. (40).

### 4.4. Free vibration of cable with one end fixed- one end mass

#### 4.4.1. Analytical solution

As it is shown in Fig.11, the cable is pinned at right and connected to mass at left. So boundary conditions would be:

\[
P \frac{\partial w}{\partial x} (0, t) = m \frac{\partial^2 w}{\partial t^2} (0, t) , \quad w(l, t) = 0
\]

Or

\[
P \frac{\partial y}{\partial x} (0, t) = -m \alpha^2 y(0) , \quad y(l, t) = 0
\]
Applying boundary conditions to Eq. (11), gives the frequency equation as:

\[
\tan \frac{\omega}{c} \left[ \frac{P}{mc\omega} \right] = 0
\]  

(45)

4.4.2. ADM solution

According to Fig. 12, dimensionless boundary conditions are:

\[
Y(l)=0 \quad \text{and} \quad P Y''(l) = -m \omega^2 Y(l)
\]  

(46)

Applying Eq. (66) to Eqs. (25) and (26), gives matrix form of governing equations as follow:

\[
\begin{bmatrix}
\sum_{k=2}^{+\infty} \left( -\eta \right)^j \omega^2 \\
\sum_{k=0}^{+\infty} \left( -\eta \right)^j \omega^2 \\
\sum_{k=2}^{+\infty} \left( -\eta \right)^j \omega^2 P
\end{bmatrix}
\begin{bmatrix}
\psi(\alpha) \\
\varphi(\alpha) \\
\beta(\alpha)
\end{bmatrix}
= 0
\]  

(47)

Assuming \( m=10 \text{kg} \) and scalar parameters of the example 1, the natural frequencies and three first mode shapes are gained. The results are presented in Table 4 and Fig. 13, respectively.

Table 4. Natural frequencies of the cable with one end fixed-one end mass

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \omega_1 ) (rad/s)</th>
<th>( \omega_2 ) (rad/s)</th>
<th>( \omega_3 ) (rad/s)</th>
<th>( \omega_4 ) (rad/s)</th>
<th>( \omega_5 ) (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>12.1673</td>
<td>48.9283</td>
<td>89.5102</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12.1669</td>
<td>48.4294</td>
<td>89.4344</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>12.1669</td>
<td>48.4455</td>
<td>91.0371</td>
<td>134.8382</td>
<td>182.4680</td>
</tr>
<tr>
<td>12</td>
<td>12.1669</td>
<td>48.4455</td>
<td>91.0371</td>
<td>134.7651</td>
<td>178.8315</td>
</tr>
<tr>
<td>16</td>
<td>12.1669</td>
<td>48.4455</td>
<td>91.0371</td>
<td>134.7651</td>
<td>178.8313</td>
</tr>
<tr>
<td>approximate value</td>
<td>12.16695</td>
<td>48.4455</td>
<td>91.0371</td>
<td>134.7651</td>
<td>178.8313</td>
</tr>
</tbody>
</table>

Figure 11. The cable with one end pinned-one end mass used in analytical solution

Figure 12. The cable with one pinned-one end mass used in ADM solution

Figure 13. Three first mode shapes of the cable with one end pinned-one end mass
4.5. Free vibration of a cable with one end free-one end mass

4.5.1. Analytical solution

As it is shown in Fig. 14, the cable is free at left and connected to mass at right. So boundary conditions would be:

\[ \frac{\partial^2 w(l, t)}{\partial x^2} = 0 \quad P \frac{\partial w(l, t)}{\partial x} = -m \frac{\partial^2 y(l, t)}{\partial x^2} \]  \hspace{1cm} (48)

Or

\[ \frac{dy(0)}{dx} = 0 \quad P \frac{dy(l)}{dx} = m \omega^2 y(l) \]  \hspace{1cm} (49)

Figure 14. The cable with one end free-one end mass used in analytical solution

Applying boundary conditions to Eq. (11), gives the frequency equation as:

\[ \tan \frac{\phi}{c} = \frac{m \omega}{P} \]  \hspace{1cm} (50)

4.5.2. ADM solution

According to Fig. 15, Dimensionless boundary conditions are:

\[ PY''(l) = m \omega^2 Y(l), \quad Y(0) = 0 \]  \hspace{1cm} (51)

Figure 15. The cable with one end free-one end mass used in ADM solution

Applying Eq. (51) to Eqs. (25) and (26) yields:

\[ \begin{bmatrix}
\sum_{i=1}^{\infty} \frac{(-\eta)^i m \omega^{2i}}{(2k-1)!} P
+ \sum_{i=0}^{\infty} \frac{(-\eta)^i \omega^{2i}}{(2k)!} P
- \sum_{i=0}^{\infty} \frac{(-\eta)^i m \omega^{2i}}{(2k+1)!} P
+ \sum_{i=0}^{\infty} \frac{(-\eta)^i \omega^{2i}}{(2k)!} P
\end{bmatrix}
\begin{bmatrix}
Y(0)

Y_{\text{in}}(0)
\end{bmatrix}
= \begin{bmatrix}
0

0
\end{bmatrix} \hspace{1cm} (52)

Assuming \( m=10 \text{kg} \) and scalar parameters of the example 1, the natural frequencies and three first mode shapes are obtained. The results are presented in table 5 and Fig. 16, respectively.

Figure 16. Three first mode shapes of the cable with one end free-one end mass
5. Investigation of the effect of the spring stiffness and mass value at the boundary conditions on the value and variation of natural frequencies

5.1. Investigation of the effect of spring stiffness

5.1.1. Cable with both ends sprung

Consider a cable with both ends sprung as shown in Fig. 17 and suppose that \( K = \frac{K_1}{2} \). For different values of \( K \), the natural frequencies of the system is calculated and presented in Table 6. Other scalar parameters are imagined identical to those provided in section 4.

![Figure 17. The cable with both ends sprung](image)

Table 6. Natural frequencies of the both ends sprung cable for different values of spring stiffness

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \omega_1 ) (rad/s)</th>
<th>( \omega_2 ) (rad/s)</th>
<th>( \omega_3 ) (rad/s)</th>
<th>( \omega_4 ) (rad/s)</th>
<th>( \omega_5 ) (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free-Free</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7153</td>
<td>222.1441</td>
</tr>
<tr>
<td>1e-4</td>
<td>0.0045</td>
<td>0.00447</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
</tr>
<tr>
<td>1e-3</td>
<td>0.0141</td>
<td>0.0141</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
</tr>
<tr>
<td>1e-2</td>
<td>0.0447</td>
<td>0.0447</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
</tr>
<tr>
<td>1e1</td>
<td>1.4142</td>
<td>1.4136</td>
<td>44.4737</td>
<td>88.8801</td>
<td>133.3014</td>
</tr>
<tr>
<td>1e2</td>
<td>4.4721</td>
<td>4.4535</td>
<td>44.4784</td>
<td>89.0821</td>
<td>133.4363</td>
</tr>
<tr>
<td>1e3</td>
<td>14.1421</td>
<td>13.5791</td>
<td>48.5218</td>
<td>91.0498</td>
<td>134.7691</td>
</tr>
<tr>
<td>1e4</td>
<td>44.7214</td>
<td>32.3070</td>
<td>67.3347</td>
<td>105.5523</td>
<td>146.0403</td>
</tr>
<tr>
<td>1e5</td>
<td>141.4214</td>
<td>42.7220</td>
<td>85.4559</td>
<td>128.2130</td>
<td>171.0040</td>
</tr>
<tr>
<td>1e6</td>
<td>447.2136</td>
<td>44.2518</td>
<td>88.5036</td>
<td>132.7555</td>
<td>177.0074</td>
</tr>
<tr>
<td>1e7</td>
<td>1.4142e+3</td>
<td>44.1140</td>
<td>88.8821</td>
<td>133.2331</td>
<td>177.6442</td>
</tr>
<tr>
<td>1e8</td>
<td>4.4721e+3</td>
<td>44.4270</td>
<td>88.8541</td>
<td>133.2811</td>
<td>177.7082</td>
</tr>
<tr>
<td>1e9</td>
<td>1.4142e+4</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7153</td>
</tr>
<tr>
<td>1e10</td>
<td>4.4721e+4</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7153</td>
<td>222.1441</td>
</tr>
</tbody>
</table>

In the second column, the first natural frequency is obtained from Eq. (53).

\[
\omega_1 = \sqrt{\frac{K_1}{m}}
\]  

In which \( K_0 = K_1 + K_2 \) and \( m \) is the mass of the cable.

As it is evident from table 6, for \( K_0 \times 10^3 \) or \( \omega_1^* \times 10^3 \), Eq. (53) can be used for estimation of the first natural frequency. Also the \( i \)th natural frequency of the system is equal to the \( (i-1) \)th natural frequency of the cable with both ends free, for which the exact solution is available \( (i \geq 2) \).

Also for \( K_0 > 10^3 \) or \( \omega_1^* > 10 \), a both ends fixed cable can be assumed instead for achieving the natural frequencies. Fig. 18 shows the value of the first natural frequency obtained using ADM and the one obtained from Eq. (53) as a function of spring stiffness in logarithmic scale. Fig. 19 shows the variation of the second, third, fourth and the fifth natural frequencies as a function of spring stiffness.
Figure 18. Comparison of the first natural frequency obtained from Eq. (53) and the ADM as a function of spring stiffness for the cable with both ends sprung

Figure 19. Variation of the second, third, fourth and the fifth natural frequencies as a function of spring stiffness for the cable with both ends sprung

5.1.2. Cable with one end fixed - one end sprung

Similar to the previous section, natural frequencies of the system is obtained as a function of spring stiffness and presented in table 7.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
<th>$\omega_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-Free</td>
<td>22.2144</td>
<td>66.6432</td>
<td>111.0720</td>
<td>155.5009</td>
<td>199.9297</td>
</tr>
<tr>
<td>1e-2</td>
<td>22.2144</td>
<td>66.6432</td>
<td>111.0720</td>
<td>155.5009</td>
<td>199.9297</td>
</tr>
<tr>
<td>1e-1</td>
<td>22.2148</td>
<td>66.6433</td>
<td>111.0721</td>
<td>155.5015</td>
<td>199.9302</td>
</tr>
<tr>
<td>1</td>
<td>22.2189</td>
<td>66.6447</td>
<td>111.0729</td>
<td>155.5015</td>
<td>199.9302</td>
</tr>
<tr>
<td>1e1</td>
<td>22.2593</td>
<td>66.6582</td>
<td>111.0810</td>
<td>155.5073</td>
<td>199.9347</td>
</tr>
<tr>
<td>1e2</td>
<td>22.6556</td>
<td>66.7929</td>
<td>111.1620</td>
<td>155.5651</td>
<td>199.9797</td>
</tr>
<tr>
<td>1e3</td>
<td>25.9734</td>
<td>68.1062</td>
<td>111.9640</td>
<td>156.1409</td>
<td>200.4284</td>
</tr>
<tr>
<td>1e4</td>
<td>37.5284</td>
<td>77.1362</td>
<td>118.6715</td>
<td>161.3423</td>
<td>204.6348</td>
</tr>
<tr>
<td>1e5</td>
<td>43.5587</td>
<td>87.1329</td>
<td>130.7016</td>
<td>174.2975</td>
<td>217.9164</td>
</tr>
<tr>
<td>1e6</td>
<td>44.3401</td>
<td>88.6803</td>
<td>133.0204</td>
<td>177.7152</td>
<td>221.7008</td>
</tr>
<tr>
<td>1e7</td>
<td>44.4199</td>
<td>88.8398</td>
<td>133.2598</td>
<td>177.6797</td>
<td>222.0997</td>
</tr>
<tr>
<td>1e8</td>
<td>44.4279</td>
<td>88.8558</td>
<td>133.2838</td>
<td>177.7117</td>
<td>222.1397</td>
</tr>
<tr>
<td>1e9</td>
<td>44.4287</td>
<td>88.8574</td>
<td>133.2862</td>
<td>177.7149</td>
<td>222.1437</td>
</tr>
<tr>
<td>1e10</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7152</td>
<td>222.1441</td>
</tr>
<tr>
<td>Fixed-Fixed</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7153</td>
<td>222.1441</td>
</tr>
</tbody>
</table>

As it is evident from table 7, for $K < 10^3$ or $\omega^*_n < 10$, a cable with one end fixed-one end free can be assumed for calculating the natural frequencies of the system. Also for $K > 10^5$ or $\omega^*_n > 100$, the natural frequencies of the system are almost equal to the natural frequencies of the cable with both ends fixed. Figure 20, shows the variation of the first natural frequency and Figure 21 shows the variation of the second, third, fourth and the fifth natural frequencies as a function of spring stiffness.
5.2. Investigation of the effects of mass value

5.2.1. Cable with one end fixed- one end mass

Similar to Fig. 12 in section 4, a cable with one end fixed- one end mass is considered. For different values of mass, the natural frequencies of the system is obtained and presented in table 8. In the second column of the table, the first natural frequency is calculated according to the Eq. (53), in which \( K_{eq} = \frac{P}{I} \). As it is obvious from the table 8, for \( m > 10^3 \) or \( \omega_1 < 14 \), Eq. (53) can be employed for calculating the first natural frequency. Also the \( i\)th natural frequency of the system is equal to \((i-1)th\) natural frequency of the cable with both ends fixed which the exact solution is available \((i \geq 2)\). Also for \( m < 1 \) or \( \omega_1 > 44 \), a free-fixed condition can be considered instead.

Table 8. Natural frequencies of the cable with one end fixed- one end mass for different values of mass

<table>
<thead>
<tr>
<th>m</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
<th>( \omega_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed-Free</td>
<td>22.2144</td>
<td>66.6432</td>
<td>111.0721</td>
<td>155.5009</td>
<td>199.9297</td>
</tr>
<tr>
<td>1e-4</td>
<td>4.4721e+3</td>
<td>22.2142</td>
<td>66.6425</td>
<td>111.0709</td>
<td>155.4993</td>
</tr>
<tr>
<td>1e-3</td>
<td>1.4142e+3</td>
<td>22.2121</td>
<td>66.6365</td>
<td>111.0609</td>
<td>155.4853</td>
</tr>
<tr>
<td>1e-2</td>
<td>447.136</td>
<td>22.1922</td>
<td>66.5766</td>
<td>110.9611</td>
<td>155.3455</td>
</tr>
<tr>
<td>1e-1</td>
<td>141.4214</td>
<td>21.9944</td>
<td>65.9838</td>
<td>109.9745</td>
<td>153.9672</td>
</tr>
<tr>
<td>1</td>
<td>44.7214</td>
<td>20.2072</td>
<td>60.8932</td>
<td>102.2209</td>
<td>144.2535</td>
</tr>
<tr>
<td>1e1</td>
<td>14.1421</td>
<td>12.1669</td>
<td>48.4455</td>
<td>91.0371</td>
<td>134.7651</td>
</tr>
<tr>
<td>1e2</td>
<td>4.4721</td>
<td>4.3989</td>
<td>44.8743</td>
<td>89.0821</td>
<td>133.4363</td>
</tr>
<tr>
<td>1e3</td>
<td>1.4142</td>
<td>1.4118</td>
<td>44.4738</td>
<td>88.8801</td>
<td>133.3014</td>
</tr>
<tr>
<td>1e4</td>
<td>0.4472</td>
<td>0.4471</td>
<td>44.4333</td>
<td>88.8599</td>
<td>133.2879</td>
</tr>
<tr>
<td>1e5</td>
<td>0.1414</td>
<td>0.1414</td>
<td>44.4292</td>
<td>88.8578</td>
<td>133.2866</td>
</tr>
<tr>
<td>1e6</td>
<td>0.0447</td>
<td>0.0447</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2865</td>
</tr>
<tr>
<td>1e7</td>
<td>0.0141</td>
<td>0.0141</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
</tr>
<tr>
<td>Fixed-Fixed</td>
<td>44.4288</td>
<td>88.8576</td>
<td>133.2864</td>
<td>177.7153</td>
<td>222.1441</td>
</tr>
</tbody>
</table>

Figure 20. Variation of the first natural frequency as a function of spring stiffness for the cable with one end fixed- one end sprung

Figure 21. Variation of the second, third, fourth and the fifth natural frequencies as a function of spring stiffness for the cable with one end fixed- one end sprung
Figure 22. compares the value of the first natural frequency obtained using ADM and obtained from Eq. (53) as a function of mass in logarithmic scale. Figure 23. demonstrates the variation of the second, third, fourth and the fifth natural frequencies as a function of mass.

![Figure 22. Comparison of the first natural frequency obtained from Eq. (53) and the ADM as a function of mass for the cable with one end fixed-one end mass](image)

![Figure 23. Variation of the second, third, fourth and the fifth natural frequencies as a function of mass for the cable with one end fixed-one end mass](image)

6. RESULTS AND DISCUSSION

As it was shown in section 4, the series solution of the ADM converges to the exact value rapidly. Any natural frequencies and corresponding mode shapes would achieved by considering sufficient number of terms of the series solution.

Having analyzed the data in section 5, it was disclosed that for a large range of mass or spring stiffness, a fixed or free condition can be assumed instead. Also the approximate relation (53), generally gives an accurate result for the first natural frequency.

As examples, in the case of the cable with both ends sprung, for $\omega^* < 14$, the Eq. (53) can be used for calculating the first natural frequency. Also the second, third, fourth and the fifth natural frequencies, are the first, second, third and the fourth natural frequencies of the cable with both ends free respectively.

In the case of the cable with one end fixed-one end sprung, for $\omega^* < 10$, the natural frequencies can be approximated by the natural frequencies of the cable with one end fixed-one end free. Also for $\omega^* > 100$, the spring condition can be replaced by a fixed one.

For the cable with one end fixed-one end mass, Eq. (53) gives good results for $\omega^* < 14$. Also the second, third, fourth and the fifth natural frequencies are approximately equal to the first, second, third and the fourth natural frequencies of the cable with both ends fixed respectively. Also for $\omega^* > 44$, a free-fixed condition can be assumed instead.
7. Conclusion

A cable with five different boundary conditions was analyzed by applying Adomian decomposition method. Using ADM, the governing differential equation is reduced to a recursive algebraic equation and the boundary conditions became simple algebraic frequency equations which are proper for symbolic computation. Any natural frequencies and corresponding mode shapes were obtained simultaneously by some simple mathematical operations.

This paper proposed an efficient procedure for analyzing vibration of cables with different boundary conditions. By increasing the number of terms, we got more natural frequencies and consequently more mode shapes with higher accuracy. The results with ADM are in excellent agreement with the analytical ones. However, for some cases, obtaining the natural frequencies from the frequency equation was not possible explicitly. In these cases the results were validated by satisfying the corresponding frequency equation.

It was revealed that if the amount of mass or spring stiffness considered in boundary conditions, gets an upper value from a limit, a fixed condition can be replaced instead. Also if it gets less, a free one can be considered instead.

Less computation effort and rapid convergence, in comparison with numerical methods, demonstrated that the ADM is a reliable and efficient procedure for solving physical differential equations.

It should be noted that the proposed method for the cable vibration analysis is generally applicable to arbitrary boundary conditions. On the basis of these results, the frequency equations derived in the present paper can be used in the design of cables with various boundary conditions.

- The authors have declared no conflict of interest.

8. REFERENCES


