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Generalized Derivations with Central Values on Lie Ideals

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ABSTRACT

Let R be a prime ring of $charR\neq 2$, H a generalized derivation and L a non-central lie ideal of R. We show that if $l^s H(l) l^t \in Z(R)$ for all $l\in L$, where s, $t\geq 0$ are fixed integers, then $H(x)=bx$ for some $b\in C$, the extended centroid of R, or R satisfies S_4. Moreover, let R be a 2-torsion free semiprime ring, let $A=O(R)$ be an orthogonal completion of R and $B=B(C)$ the Boolean ring of C. Suppose $[x_1,x_2]^s H([x_1,x_2])[x_1,x_2]^t \in Z(R)$ for all $x_1,x_2 \in R$, where $s,t\geq 0$ are fixed integers. Then there exists idempotent $e\in B$ such that $H(x)=bx$ on eA and the ring $(1-e)A$ satisfies S_4.

MSC: 16R50; 16N60; 16D60
KEY WORDS: generalized derivation, prime ring, Martindale quotient ring.

1. INTRODUCTION

Let R be an associative ring with center $Z(R)$. Recall that an additive map $d:R\rightarrow R$ is called derivation if $d(xy)=d(x)y+xd(y)$, for all $x,y\in R$. Many results in literature indicate that global structure of a prime (semiprime) ring R is often lightly connected to the behavior of additive mappings defined on R. A well-known result of Herstein [6] stated that if d is a nonzero derivation of a prime ring R such that $d(x)^n\in Z(R)$ for all $x\in R$, then R satisfies S_4, the standard identity in four variables. Herstein's result was extended to the case of Lie ideals of prime rings by Bergen and Carini [2]. Some articles was studied derivation with central values on Lie ideals [4, 10]. Recently, Dhara [5] studied the more generalized situation when $l^s d(l) l^t \in Z(R)$, for all $l\in L$, the non-central Lie ideal of R, where $s,t\geq 0$ are some fixed integers.

Here we will consider the same situation in case the derivation d is replaced by generalized derivation H. More specifically an additive map $H:R\rightarrow R$ is called generalized derivation if there is a derivation d of R such that $H(xy)=H(x)y+xd(y)$, for all $x,y\in R$.
Throughout the paper we use the standard notation from [1]. In particular, we denote by Q the two sided Martindale quotient of prime (semiprime) ring R and C the center of Q. We call C the extended centroid of R.
The main results of this paper are as follows:

Theorem 1.1. Let R be a prime ring of $charR\neq 2$, H generalized derivation and L a non-central Lie ideal of R. Suppose $l^s H(l) l^t \in Z(R)$ for all $l\in L$, where $s,t\geq 0$, are fixed integers. Then $H(x)=bx$ for some $b\in C$, the extended centroid of R, or R satisfies S_4.

When R is a semiprime ring, we prove:
Theorem 1.2. let R be a 2-torsion free semiprime ring with generalized derivation H. Consider $[x_1,x_2]^s H([x_1,x_2])[x_1,x_2]^t \in Z(R)$ for all $x_1,x_2\in R$, where $s,t\geq 0$ are fixed integers. Further, let $A=O(R)$ be the orthogonal completion of R and $B=B(C)$ where C is the extended centroid of R. Then there exists idempotent $e\in B$ such that $H(x)=bx$ on eA and the ring $(1-e)A$ satisfies S_4.

2. PROOF OF THE MAIN RESULT

The following results are useful tools needed in the proof of the main results.

Lemma 2.1. Every generalized derivation H on a dense right ideal of prime (semiprime) ring R can be uniquely extended to a generalized derivations of Q. Also can be write in the form $H(x)=bx+d(x)$ for some $b\in Q$, all $x\in Q$ and a derivation d of Q [11].
Lemma 2.2. (see [8, Lemma 2] and [3, Lemma 1]). Let R be a prime ring of $charR\neq 2$, L be a non-central Lie ideal of R and I be the ideal of R generated by $[L,L]$. Then $I\subseteq L+L^2$ and $[I,I]\subseteq L$.
Theorem 2.3. (Kharchenko [7]). Let R be a prime ring, d a nonzero derivation of R and I a nonzero ideal of R. If I

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satisfies the differential identity

$$f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n))=0,$$

for any $r_1, r_2, \ldots, r_n \in I$, then one of the following holds:

(i) I satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0.$$

(ii) d is Q-inner, that is, for some $q\in Q$, $d(x)=[q, x]$ and I satisfies the generalized polynomial identity

$$f(r_1, r_2, \ldots, r_n, [q,r_1], [q,r_2], \ldots, [q,r_n])=0.$$

We establish the following technical results required in the proof of Theorem 1.1.

Lemma 2.4. Let $R=M_k(F)$ be a ring of all $k\times k$ matrices over a field F where $k \geq 3$. Suppose $b[x_1, x_2]+[x_1, x_2]c\in Z(R)$ for some $b,c\in R$ and all $x_1, x_2\in R$. Then $b, c\in F.I_k$.

Proof. Let $b=(b_{ij})_{k\times k}$, $c=(c_{ij})_{k\times k}$. Putting $x_1=e_{11}$ and $x_2=e_{12}$, we obtain $b[x_1,x_2]+[x_1,x_2]c =be_{12}+e_{12}c$. Since rank of $b[x_1, x_2]+[x_1, x_2]c \leq 2$, it cannot be invertible. This implies $be_{12}+e_{12}c=0$. Left and right multiplying by e_{12}, we get

$$0=e_{12}(be_{12}+ e_{12}c)=b_{21}e_{12},$$
$$0 = (be_{12}+e_{12}c)e_{12}= c_{21}e_{12}.$$

This implies that $c_{21}=b_{21}=0$. Thus for any $i\neq j$, $b_{ij}=c_{ij}=0$. That is, b and c are diagonal. Let $b=\sum_{i=1}^{k} b_{ii}e_{ii}$, for any F-automorphism θ of R b^θ enjoys the same property as b does, namely, $b^\theta[x_1,x_2]+[x_1,x_2]c^\theta$ is zero or invertible, for every $x_1,x_2\in R$. Hence b^θ must be diagonal. Then for each $j\neq 1$,

$$(1+e_{1j})b(1-e_{1j}) = \sum_{i=1}^{k} b_{ii}e_{ii} + (b_{jj} - b_{11})e_{1j} ,$$

is diagonal. Therefore, $b_{jj}=b_{11}$ and so $b\in F.I_k$. Similarly, we conclude $c\in F.I_k$. □

Lemma 2.5. Let $R=M_k(F)$ be a ring of all $k\times k$ matrices over a field F of $charF\neq 2$, where $k \geq 3$. Suppose $[x_1,x_2]^s(b[x_1,x_2]+[x_1,x_2])[x_1,x_2]^t \in Z(R)$, for some $b,c\in R$ and all $x_1, x_2\in R$ where $s,t\geq 0$ are fixed integers such that $s+t\neq 0$. Then $b, c\in F\cdot I_k$.

Proof. Let $b=(b_{ij})_{k\times k}$, $c=(c_{ij})_{k\times k}$ and set

$$f(x_1, x_2)=[x_1, x_2]^s(b[x_1, x_2]+[x_1,x_2]c)[x_1, x_2]^t.$$

Putting $x_1=e_{11}$, $x_2=e_{12}-e_{21}$, we obtain $[x_1, x_2]=e_{12}+e_{21}$ and $[x_1, x_2]^n=e_{11}+e_{22}$ for $n\geq 2$. So we have four cases:

Case 1. $s=t=1$. We get

$$f(x_1, x_2)=(b_{21}+c_{12})e_{11}+(b_{12}+c_{21})e_{22}+(b_{22}+c_{11})e_{12}+ (b_{11}+c_{22})e_{21}.$$

Case 2. $s=0$ and $t=1$. We get

$$f(x_1,x_2)=(b_{11}+c_{22})e_{11}+(b_{22}+c_{11})e_{22}+(b_{12}+c_{21})e_{12}+(b_{21}+c_{12})e_{21}+\sum_{i=3}^{k} b_{i1}e_{i1} + \sum_{i=3}^{k} b_{i2}e_{i2}$$

Case 3. $s= 1$ and $t= 0$. We get

$$f(x_1,x_2)=(b_{22}+c_{11})e_{11}+(b_{11}+c_{22})e_{22}+(b_{21}+c_{12})e_{12}+(b_{12}+c_{21})e_{21}+\sum_{i=3}^{k} c_{1i}e_{1i} + \sum_{i=3}^{k} c_{2i}e_{2i}$$

Case 4. $s,t\geq 2$. We obtain

$$f(x_1,x_2)=(b_{12}+c_{21})e_{11}+(b_{21}+c_{12})e_{22}+(b_{11}+c_{22})e_{12}+(b_{22}+c_{11})e_{21}.$$

In each case, since rank of $f(x_1,x_2)\leq 2$, $f(x_1, x_2) = 0$. Thus

$$b_{12}=-c_{21} \text{ and } b_{21}=-c_{12},$$

and so for any $i\neq j$ we have

(1) $$b_{ij}= -c_{ji}$$

Now putting $x_1=e_{11}$, $x_2=e_{12}+e_{21}$, we have $[x_1,x_2]^n=(-1)^{\frac{n}{2}}(e_{11}+ e_{22})$ if n is even and $(-1)^{\frac{n-1}{2}}(e_{12}-e_{21})$ if n is odd.

Four cases may be occurred:

Case 1. s and t are even. We get

$$f(x_1, x_2) = \pm((-b_{12}+ c_{21})e_{11}+ (b_{21}-c_{12})e_{22}+ (b_{11}+c_{22})e_{12}+ (-b_{22}-c_{11})e_{21}).$$

Case 2. s and t are odd. We get

$$f(x_1, x_2) = \pm((-b_{21}+ c_{12})e_{11}+ (b_{12}- c_{21})e_{22}+ (-b_{22}- c_{11})e_{12}+ (b_{11}+ c_{22})e_{21}).$$

Case 3. s is even and t is odd. We get

$$f(x_1, x_2) = \pm((-b_{11}-c_{22})e_{11}+ (-b_{22}-c_{11})e_{22}+ (-b_{12}+c_{21})e_{12}+ (-b_{21}+c_{12})e_{21}).$$

Case 4. s is odd and t is even. We get

$$f(x_1, x_2) = \pm((-b_{22}- c_{11})e_{11}+ (-b_{11}- c_{22})e_{22}+ (b_{21}- c_{12})e_{12}+ (b_{12}- c_{21})e_{21}).$$

In each cases, since rank of $f(x_1, x_2) \leq 2$, $f(x_1, x_2) = 0$. Thus

$$b_{12} = c_{21} \quad \text{and} \quad b_{21} = c_{12},$$

and so for any $i \neq j$ we have

(2) $$b_{ij} = c_{ji}.$$

(1) and (2) imply that b and c are diagonal. So we apply the same argument used in the proof of Lemma 2.4. Hence $b, c \in F \cdot I_k$.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Since $charR \neq 2$ and L is non-central Lie ideal, by Lemma 2.2 there exists an ideal I of R such that $0 \neq [I, I] \subseteq L$ and $[L,L] \neq 0$. Hence, without loss of generality, we may assume $L = [I, I]$. Thus I satisfies the generalized differential identity

$$[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t \in Z(R).$$

Let Q be the two sided Martindale quotient ring and C the extended centroid of R. By [11] I and Q satisfy the same differential identities, thus we may assume

$$[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t \in Z(R),$$

for all $x_1, x_2 \in Q$. By Lemma 2.1 we may assume $H(x) = bx + d(x)$ for some $b \in Q$ all $x \in Q$ and d a derivation of Q. Hence Q satisfies

$$[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t \in Z(R).$$

This is a polynomial identity. Hence there exists a field F such that $Q \subseteq M_k(F)$, the ring of $k \times k$ matrices over field F, where $k > 1$. Moreover Q and $M_k(F)$ satisfy the same polynomial identity [9]. Hence we have

(3) $$[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t \in Z(M_k(F)).$$

Now consider two cases:

case 1. d is a Q-inner derivation. In this case, there exists an element $p \in Q$ such that $d(x) = [p, x]$ for all $x \in M_k(F)$, then (3) becomes

$$[x_1, x_2]^s (b[x_1, x_2] + [p, [x_1, x_2]])[x_1, x_2]^t \in Z(M_k(F)).$$

So

$$[x_1, x_2]^s ((b+p)[x_1, x_2] - [x_1, x_2]p)[x_1, x_2]^t \in Z(M_k(F)),$$

for all $x_1, x_2 \in M_k(F)$. In this case if $k \geq 3$ and $s = t = 0$, then by Lemma 2.4 we have $-p$, $b+p \in F.I_k$. Also for $k \geq 3$ and $s+t \neq 0$, Lemma 2.5 implies $-p$, $b+p \in F.I_k$. Then $b \in F.I_k$, and so $d(x) = 0$. Hence $H(x) = bx$ for all $x \in M_k(F)$. So by [9] for all $x \in R$ we have $H(x) = bx$. If $k = 2$, then R satisfies S_4.

case 2. d is not a Q-inner derivation. In this case we have

$$[[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t, x_3] = 0,$$

for all $x_1, x_2, x_3 \in M_k(F)$.

Then by Theorem 2.3 we have

$$[[x_1, x_2]^s (b[x_1, x_2] + [x_4, x_2] + [x_1, x_5])[x_1, x_2]^t, x_3] = 0,$$

for all $x_1, x_2, x_3, x_4, x_5 \in M_k(F)$. In particular, $M_k(F)$ satisfies its blended component

$$[[x_1, x_2]^s ([x_4, x_2] + [x_1, x_5])[x_1, x_2]^t, x_3] = 0.$$

If $k \geq 3$, then by choosing

$$x_1 = e_{ij}, \ x_2 = e_{ji}, \quad x_3 = e_{ik}, \quad x_4 = e_{ij}, \quad x_5 = 0,$$

For all $i \neq j \neq k$, we get

$$0 = [[x_1, x_2]^s ([x_4, x_2] + [x_1, x_5])[x_1, x_2]^t, x_3] = e_{ik},$$

which is a contradiction. Thus $k = 2$, that is, R satisfies S_4.

Now let R be a semiprime orthogonally complete ring with extended centeroid C. The notations $B = B(C)$ and $spec(B)$ denotes Boolian ring of C and the set of all maximal ideal of B, respectively. It is well known that if $M \in spec(B)$ then $R_M = R/RM$ is prime [1,Theorem 3.2.7]. We use the notations Ω-Δ-ring, Horn formulas and Hereditary formulas. We refer the reader to [1, pages 37, 38, 43, 120] for the definitions and the related properties of these objects.

We establish the following technical result required in the proof of Theorem 1.2.

Lemma 2.6. [1, Theorem 3.2.18]. Let R be an orthogonally complete Ω-Δ-ring with extended centroid C, $\Psi_i(x_1, x_2, \ldots, x_n)$ Horn formulas of signature Ω-Δ, $i = 1, 2, \ldots$ and $\Phi(y_1, y_2, \ldots, y_m)$ a Hereditary first order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \ldots, a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, \ldots, c_m) \in R^{(m)}$. Suppose $R \vDash \phi(\vec{c})$ and for every $M \in spec(B)$ there exists a natural number $i = i(M) > 0$ such that

$$R_M \vDash \Phi(\phi_M(\vec{c})) \Rightarrow \Psi_i(\phi_M(\vec{a})),$$

Where $\phi_M: R \to R_M = R/RM$ is the canonical projection. Then there exists a natural number $k>0$ and pairwise orthogonal idempotents $e_1, e_2, \ldots, e_k \in B$ such that $e_1 + e_2 + \ldots + e_k = 1$ and $e_i R \vDash \Psi_i(e_i \vec{a})$ for all $e_i \neq 0$.

We denote $O(R)$ the orthogonal completion of R which is defined as the intersection of all orthogonally complete subset of Q containing R.
Now we can prove Theorem 1.2.

Proof of Theorem 1.2. By assumption we have R satisfies

$$[[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t, x_3] = 0.$$

By Lemma 2.1, the generalized derivation H can be extended uniquely to the generalized derivation on Q, moreover, we may assume $H([x_1, x_2]) = b[x_1, x_2] + d([x_1, x_2])$, for some $b \in Q$, all $x_1, x_2 \in Q$ and d a derivation of Q. Hence Q satisfies

$$[[x_1, x_2]^s (b([x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t, x_3] = 0.$$

According to [1, Theorem 3.1.16] $d(A) \subseteq A$ and $d(e)=0$ for all $e \in B$. Therefore, A is an orthogonally complete Ω-Δ-ring, where $\Omega = \{o, +, -, \cdot, d\}$. Consider formulas

$$\Phi = (\forall x_1)(\forall x_2)\|[[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2])[x_1, x_2]^t, x_3] = 0\|,$$

$$\Psi_1 = (\forall x)\|H(x) = bx\|,$$

$$\Psi_2 = (\forall x_1)(\forall x_2)(\forall x_3)(\forall x_4)\|S_4(x_1, x_2, x_3, x_4) = 0\|.$$

We can easily check that Φ is a hereditary first order formula and $\neg\Phi$, Ψ_1, Ψ_2 are Horn formulas. So using Theorem 1.1, all conditions of Lemma 2.6 are fulfilled. Hence there exist two orthogonal idempotents e_1 and e_2 such that $e_1 + e_2 = 1$. If $e_i \neq 0$, then $e_i A \vDash \Psi_i$, $i = 1,2$. This completes the proof.

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