# A Posteriori Error for The Telegraph Equation with A Local Damping in Mobile Phones 

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#### Abstract

. We study a finite element method applied to a system of linear telegraph equations in a bounded smooth domain in $\mathrm{R}^{d}, d=1,2,3$, associated with a locally distributed damping function. We start with a spatially continuous finite element formulation allowing jump discontinuities in time. This approach yields, $L_{2}\left(L_{2}\right)$ and $L_{\infty}\left(L_{2}\right)$, a posteriori error estimates in terms of weighted residuals of the system. The proof of the a posteriori error estimates is base on the strong stability estimates for the corresponding adjoint equations. Optimal convergence rates are derived upon the maximal available regularity of the exact solution. KEYWORDS: Discontinuous Galerkin method, hyperbolic problem, telegraph equation, finite element, a posteriori error estimate, streamline diffusion method.


## 1 INTRODUCTION

We consider a system of linear multidimensional telegraph equation associated with locally damping terms. Introducing vector quantities related to a solution, we can convert this hyperbolic system as an elliptic system of equations. We also formulate a streamline diffusion method adequate for the finite element solution to the hyperbolic type partial differential equations. However, this will not be our main concern. We focus on a spatially continuous finite element scheme (with a streamline-diffusion type structure, but without the streamline diffusion term) for a new elliptic system of equations, where jump discontinuous over certain time levels are allowed. For this system, we derive a posteriori error estimates in the $L_{2}\left(L_{2}\right)$ - and $L_{\infty}\left(L_{2}\right)$-norms.

Studies of this type were consider by Gergoulus and co-workers [7], where, using the Galerkin finite element method for the linear wave equation without damping term, they obtained a posteriori error estimates in the $L_{\infty}\left(L_{2}\right)$ -norm. Johnson [11] established the existence of a solution to the second order hyperbolic problem. He used the discontinuous Galerkin method to obtain a priori and a posteriori error $L_{2}$-estimates.

We consider the following model problem which is appear electrical signals along a telegraph line, digital image processing, telecommunication, solar cell, solar radiation, signals and systems (cf. [16-23]): construct an algorithm for numerical solving a system of linear telegraph equation with energy decay such that the error between a given tolerance such that the computational work in nearly minimal. More specifically, we consider the following system of linear telegraph equations:

$$
\left\{\begin{array}{l}
u_{t t}-\Delta u+c u_{t}+\lambda u=f, \quad \text { in } \Omega \times[0, \infty],  \tag{1}\\
u=0, \quad \text { in } \partial \Omega \times[0, \infty], \\
u(x, 0)=u_{0}(x), \quad \text { in } \Omega, \\
u_{t}(x, 0)=u_{1}(x), \quad \text { in } \Omega, \quad\left(u_{t}:=\partial u / \partial t\right),
\end{array}\right.
$$

where $\Omega \in \mathrm{R}^{d}, d=1,2,3$, is a bounded domain with smooth boundary $\partial \Omega$ (for $\mathrm{d}=2,3$ ) and $c, \lambda>0$ are constants. Here, $\Delta$ denotes the Laplace operator in the spatial variable $x$.

Studies of telegraph equations were arisen of propagation of electrical signals in a cable of transmission line, wave phenomena and mobile phones [22]. Also, biologists study these equations in pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge. A maximum principle for bounded and periodic solutions of the telegraph equation, presented in [19], respectively. The existence of time-periodic solutions of the telegraph equation can be found in [17] and the references therein. The authors of [16, 18, 20] have developed by family of finite

[^0]difference methods and mesh-less methods for solving telegraph equations. But in all references, we can not observe any things about approximation solutions and error analysis for these equations. Therefore, in this paper, we try to give a new method for solving it such that it provides some optimal error bounds for this approximated method.

We propose the vector form and reformulate the system (1) as the following abstract elliptic system of partial differential equations:

$$
\left\{\begin{array}{l}
L w:=w_{t}+A w=F, \quad \operatorname{in} \Omega \times[0, \infty]  \tag{1}\\
w(0)=w_{0}, \quad \operatorname{in} \Omega
\end{array}\right.
$$

where $w(x, t)=(u(x, t), v(x, t))^{T}, v=u_{t}$ and the operator $A$ is defined by the formula

$$
A:\left[H_{0}^{1} \times L_{2}\right] \rightarrow\left[H_{0}^{1} \times L_{2}\right]
$$

with the domain

$$
D(A)=\left[\left(H_{0}^{1} \cap H^{2}\right) \times H_{0}^{1}\right]
$$

and the matrix-operator form

$$
A=\left[\begin{array}{cc}
0 & -\mathrm{l} \\
-\Delta+\lambda \mid & c \mathrm{l}
\end{array}\right]
$$

where $I$ is the identity operator. We also introduce

$$
F(x, t)=(0, f(x, t))^{T}, \quad \text { and } \quad w_{0}(x)=\left(u_{0}(x), u_{1}(x)\right)
$$

Let $L_{2}(\Omega \times[0, \infty)):=H^{0}(\Omega \times[0, \infty))$ be the usual Sobolev spaces of Lebesgue square integrable functions defined in $\Omega \times[0, \infty)$. By $H_{0}^{1}(\Omega \times[0, \infty))$, we mean a subspace of $H^{1}(\Omega \times[0, \infty))$ consisting of functions vanishing on $\partial \Omega \times[0, \infty)$, where $H_{0}^{1}(\Omega \times[0, \infty))$ consists of all functions in $H^{0}(\Omega \times[0, \infty))$ possessing all first order partial derivatives in $H^{0}(\Omega \times[0, \infty))$.

The paper is organized as follows. Section 2 contains preliminaries and a formulation of the finite element method for (1), considering space-time slabs: $S_{n}:=\Omega \times I_{n}$, where $I_{n}=\left(t_{n}, t_{n+1}\right), n=0,1,2, \ldots, N-1$ are sub-intervals of the time domain. In Subsection 3.1, we study a posteriori error estimates for (1) and derive optimal $L_{2}\left(L_{2}\right)-$ and $L_{\infty}\left(L_{2}\right)$ - norm error bounds. In Subsection 3.2, we introduce projection operators and, again using duality, derive the interpolation estimates and complete the proof of the a posteriori error bounds. In Section 4, we prove the strong stability estimates for dual problems. Finally, we give conclusion remarks in Section 5.

## 2 Notation and Preliminaries

In this section, we consider a time discontinuous Galerkin method for solving (1) which is based on the use of finite elements over the space-time domain $\Omega \times[0, T]$. To describe this method, we consider a subdivision

$$
0=t_{0}<t_{1}<\ldots<t_{N}=T,
$$

of the time interval $[0, T]$ into sub-intervals $I_{n}=\left(t_{n}, t_{n+1}\right)$, with the time steps $k_{n}=t_{n+1}-t_{n}$, $n=0,1, \ldots, N-1$ and introduce the corresponding space-time slabs

$$
\begin{equation*}
S_{n}=\left\{(x, t): \quad x \in \Omega, \quad t_{n} \leq t<t_{n+1}\right\}, \quad n=0,1, \ldots, N-1 \tag{2}
\end{equation*}
$$

For notational convenience we denote by $k=k(t)$ the mesh function for the time discretization, where $k(t)=k_{n}$ for $t \in\left(t_{n}, t_{n+1}\right)$. We also assume that $\Omega$ is a bounded open interval in the one-dimensional case an open bounded subset in $\mathrm{R}^{d}$ with a piecewise smooth boundary $\partial \Omega$ in the case $d \geq 2$. We use the standard procedure partitioning $\Omega$ into sub-intervals for $d=1$, quasiuniform triangular elements for $d=2$, or tetrahedrons (with the corresponding minimal vertex angle conditions) for $d=3$.

### 2.1 Implementation of the time discontinuous Galerkin scheme

For every $n$ we denote $\mathbf{W}^{n}$ be a finite element subspace of $\left[H_{0}^{1}\left(S_{n}\right) \times L_{2}\left(S_{n}\right)\right]$. On each slab $S_{n}$, we
formulate the following spatially continuous problem: for every $n=0, \ldots, N-1$ find $\mathbf{w}^{n} \in \mathbf{W}^{n}$ such that

$$
\begin{equation*}
\left(w_{t}^{n}+A w^{n}, g\right)_{n}+<w_{+}^{n}, g_{+}>_{n}=(F, g)_{n}+<w_{-}^{n}, g_{+}>_{n}, \quad \forall \quad g \in \mathbf{W}^{n} \tag{3}
\end{equation*}
$$

where

$$
\begin{gathered}
(\mathbf{w}, \mathbf{g})_{n}=\int_{S_{n}} \mathbf{w}^{T} \cdot \mathbf{g} d x d t, \\
<\mathbf{w}, \mathbf{g}>_{n}=\int_{\Omega} \mathbf{w}^{T}\left(x, t_{n}\right) \cdot \mathbf{g}\left(x, t_{n}\right) d x, \\
\mathbf{w}_{ \pm}(x, t)=\lim _{s \rightarrow 0^{ \pm}} \mathbf{w}(x, t+s) .
\end{gathered}
$$

The <,> -term yields a jump which imposes a weakly enforced continuity condition across the slab interfaces at each time level $t:=t_{n}$ a mechanism which governs the flow of information from one slab to adjacent one in the positive time direction. Note that we defined the inner product of $\mathbf{w}_{j}=\left(u_{j}, v_{j}\right)^{T}, j=1,2$, in the space $\left[H_{0}^{1}\left(S_{n}\right) \times L_{2}\left(S_{n}\right)\right], n=0,1, \ldots, N-1$, by the formula

$$
\begin{equation*}
\left(\mathbf{w}_{1}, \mathbf{w}_{2}\right)_{n}=\int_{S_{n}}\left(\nabla u_{1} \nabla u_{2}+v_{1} v_{2}\right) d x d t \tag{4}
\end{equation*}
$$

Summing over n , we get the function space

$$
\mathbf{W}:=\prod_{n=0}^{N-1} \mathbf{W}^{n} .
$$

Thus we can write (3) in a more concise form as follows: find $\tilde{\mathbf{w}} \in \mathbf{W}$ such that

$$
\begin{equation*}
B(\tilde{w}, g)=L(g), \quad \forall \quad g \in \mathbf{W} \tag{5}
\end{equation*}
$$

where the bilinear form $B(.,$.$) and the linear form L($.$) are defined by the formulas$

$$
\begin{equation*}
B(\tilde{w}, g)=\sum_{n=0}^{N-1}\left(\tilde{w}_{t}^{n}+A \widetilde{\mathbf{w}}^{n}, g\right)_{n}+\sum_{n=1}^{N-1}<\left[\tilde{\mathbf{w}}^{n}\right], \mathbf{g}_{+}>_{n}+<\tilde{\mathbf{w}}_{+}^{n}, \mathbf{g}_{+}>_{0} \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
L(g)=\left\langle\mathbf{w}_{0}, \mathbf{g}_{+}>_{0}+\sum_{n=0}^{N-1}(F, \mathbf{g})_{n},\right. \tag{7}
\end{equation*}
$$

respectively. The corresponding weak formulation for the continuous problem (1) is as follows:

$$
\begin{equation*}
B(w, g)=L(g), \quad \forall \quad g \in\left[H_{0}^{1}(\Omega) \times L_{2}(\Omega)\right] \tag{8}
\end{equation*}
$$

where we replace $w$ in (6) by $w$ and put the jump $[w]=0$. We set $\mathbf{w}^{n}=\left(u^{n}, v^{n}\right)^{T}$ and introduce the jump $\left[\mathbf{w}^{n}\right]=\left(\left[u^{n}\right],\left[v^{n}\right]\right)^{T}$, where $[q]=q_{+}-q_{-}$for $q=u^{n}, v^{n}$. Finally, let $\mathcal{J}_{h}$ be a partition of $\Omega$ into quasiuniform triangular $(\mathrm{d}=2)$ or tetrahedral $(\mathrm{d}=3)$ domains of the maximal diameter $h$ (the mesh size). We introduce

$$
\mathbf{W}_{h}^{n}=\left\{\mathbf{w}^{n} \in\left[H_{0}^{1}\left(S_{n}\right) \times L_{2}\left(S_{n}\right)\right]:\left.\mathbf{w}^{n}\right|_{K} \in\left[P_{l}(K) \times P_{l}(K)\right] \quad \text { for } K \in \mathfrak{J}_{h}\right\}
$$

Where $P_{l}(K)$ denotes the set of polynomials in $K$ of degree less than or equal $l$ and define the discrete function space $W_{h}$ by the formula:

$$
\begin{gather*}
W_{h}=\prod_{n=0}^{N-1} W_{h}{ }^{n} . \\
B\left(w_{h}, g\right)=L(g), \quad \forall \quad g \in \mathbf{W}_{h} . \tag{9}
\end{gather*}
$$

Finally, subtracting (9) from (8), for $\mathbf{g} \in \mathbf{W}_{h}$ we end up with the Galerkin orthogonality relation

$$
\begin{equation*}
B(e, g)=0, \quad \forall g \in \mathbf{W}_{h} \tag{10}
\end{equation*}
$$

where $e=w-w_{h}$ stands for the error.

## 2 A POSTERIORI ERROR ANALYSIS

In this section, we estimate the error of a particular approximation of solution by using the information from computation. The procedure is split in the following two subsections.

### 3.1 Dual problem, stability and error representation formula in $L_{2}\left(L_{2}\right)$

In this section, we state the dual problem for the weak (variational) formulation of the continuous problem (1) with jump discontinuities across time levels $t=t_{n}$ : find $\mathbf{W}_{h} \in \mathbf{W}_{h}$ such that for $n=0,1, \ldots, N-1$

$$
\begin{equation*}
\sum_{n=0}^{N-1}\left(\mathbf{w}_{h, t}^{n}+A \mathbf{w}_{h}^{n}, \mathbf{g}\right)_{n}+\sum_{n=1}^{N-1}<\left[\mathbf{w}_{h}^{n}\right], \mathbf{g}_{+}>_{n}+<\mathbf{w}_{h,+}^{n}, \mathbf{g}_{+}>_{0}=\sum_{n=0}^{N-1}(F, \mathbf{g})_{n}+<\mathbf{w}_{0}, \mathbf{g}_{+}>_{0} \tag{11}
\end{equation*}
$$

where $g \in \mathbf{W}_{h}$ and $\mathbf{w}_{h,-}^{0}=w_{0}$. To obtain a representation of the error, we consider the dual problem: find $\Phi \in\left[H_{0}^{1}(\Omega \times[0, \infty)) \times L_{2}(\Omega \times[0, \infty))\right]$ such that

$$
\left\{\begin{array}{rc}
L^{*} \Phi \equiv-\Phi_{t}+A^{T} \Phi=\Psi^{-1} \mathbf{e}, & \operatorname{in} \Omega  \tag{12}\\
\left.\Phi(x, t)\right|_{t=T}=0, & x \in \Omega
\end{array}\right.
$$

where $L^{*}$ denotes the adjoint of the operator $L$ defined in (1) and $\Psi$ is a positive weight function. Note that this problem is computed "backward", but there is a corresponding change in sign. We introduce the weighted $L_{2}$-norm:

$$
\begin{equation*}
\|\mathbf{u}\|_{L_{2}^{\psi}(\Omega)}=(\mathbf{u}, \Psi \mathbf{u})_{\Omega}^{1 / 2} \tag{13}
\end{equation*}
$$

Multiplying (12) by $\mathbf{e}$ and integrating by parts, over $\Omega$, yields the following error representation formula:

$$
\begin{align*}
\|\mathbf{e}\|_{L_{2}^{\Psi^{-1}}(\Omega)}^{2} & =\left(\mathbf{e}, \Psi^{-1} \mathbf{e}\right)_{\Omega}=\left(\mathbf{e}, L^{*} \Phi\right)_{\Omega} \\
& =\sum_{n=0}^{N-1}\left(\mathbf{e},-\Phi_{t}+A^{T} \Phi\right)_{n}=\sum_{n=0}^{N-1}\left(\mathbf{e},-\Phi_{t}\right)_{n}+\sum_{n=0}^{N-1}\left(\mathbf{e}, A^{T} \Phi\right)_{n} \tag{14}
\end{align*}
$$

Further partial integration in $t$ yields

$$
\begin{equation*}
\left(\mathbf{e},-\Phi_{t}\right)_{n}=-\int_{\Omega}\left(\left.\mathbf{e}^{T}(x, t) \cdot \Phi(x, t)\right|_{t=t_{n}} ^{t=t_{n+1}}\right) d x+\left(\mathbf{e}_{t}, \Phi\right)_{n} \tag{15}
\end{equation*}
$$

We define $\mathbf{e}=\mathbf{e}(x, t)=\left(e_{1}, e_{2}\right)^{T}$ and $\Phi=\Phi(x, t)=\left(\phi_{1}, \phi_{2}\right)^{T} ;$ moreover for $n=0,1, \ldots, N-1$

$$
\begin{align*}
\left(\mathbf{e}, A^{T} \Phi\right)_{n} & =\int_{S_{n}} \mathbf{e}^{T}\left[\begin{array}{cc}
0 & -\Delta+\lambda \mid \\
-\mathbf{I} & c \mathbf{|}
\end{array}\right]\left[\begin{array}{l}
\phi_{1} \\
\phi_{2}
\end{array}\right] d x d t  \tag{16}\\
& =\int_{S_{n}}\left[e_{1}, e_{2}\right]\left[\begin{array}{c}
-\Delta \phi_{2}+\lambda \phi_{2} \\
-\phi_{1}+c \phi_{2}
\end{array}\right] d x d t
\end{align*}
$$

Hence

$$
\begin{aligned}
\left(\mathbf{e}, A^{T} \Phi\right)_{n} & =\int_{S_{n}}\left(-e_{1} \Delta \phi_{2}+\lambda \phi_{2} e_{1}-\phi_{1} e_{2}+c \phi_{2} e_{2}\right) d x d t \\
& =\int_{S_{n}}\left(\nabla e_{1} \nabla \Phi_{2}+\lambda \phi_{2} e_{1}-\phi_{1} e_{2}+c \phi_{2} e_{2}\right) d x d t \\
& =\int_{S_{n}}\left(-\Delta e_{1} \phi_{2}+\lambda \phi_{2} e_{1}-\phi_{1} e_{2}+c \phi_{2} e_{2}\right) d x d t \\
& =\int_{S_{n}}(A \mathbf{e})^{T} \cdot \Phi d x d t=(A \mathbf{e}, \Phi)_{n}
\end{aligned}
$$

Now, we compute the sum of the jumps on the right-hand side of (15):

$$
\begin{aligned}
J & =\sum_{n=0}^{N-1} \int_{\Omega}\left(\mathbf{e}^{T}\left(x, t_{n+1}\right) \cdot \Phi\left(x, t_{n+1}\right)-\mathbf{e}^{T}\left(x, t_{n}\right) \cdot \Phi\left(x, t_{n}\right)\right) d x \\
& =\left(\left\langle\mathbf{e}_{-}, \Phi_{-}\right\rangle_{1}-\left\langle\mathbf{e}_{+}, \Phi_{+}\right\rangle_{0}\right)+\left(\left\langle\mathbf{e}_{-}, \Phi_{-}\right\rangle_{2}-\left\langle\mathbf{e}_{+}, \Phi_{+}\right\rangle_{1}\right)+\ldots \\
& +\left(\left\langle\mathbf{e}_{-}, \Phi_{-}\right\rangle_{N-1}-\left\langle\mathbf{e}_{+}, \Phi_{+}\right\rangle_{N-2}\right)+\left(\left\langle\mathbf{e}_{-}, \Phi_{-}\right\rangle_{N}-\left\langle\mathbf{e}_{+}, \Phi_{+}\right\rangle_{N-1}\right) .
\end{aligned}
$$

We rearrange the above sum by writing $\Phi_{-}^{n}=\Phi_{-}^{n}-\Phi_{+}^{n}+\Phi_{+}^{n}, n=1, \ldots, N-1$. Then we can write

$$
-J=<\mathbf{e}_{-}, \Phi_{-}>_{N}+<\mathbf{e}_{+}, \Phi_{+}>_{0}+\sum_{n=0}^{N-1}<[\mathbf{e}], \Phi_{+}>_{n}+\sum_{n=0}^{N-1}<\mathbf{e}_{-},[\Phi]>_{n}
$$

According to (13), $\Phi\left(\cdot, t_{N}=T\right)=0$ and since $\mathbf{e}_{-}^{0}=\left[\mathbf{w}_{0}\right]=0$, we get

$$
\begin{equation*}
J=\sum_{n=0}^{N-1}<\left[\mathbf{w}_{h}\right], \Phi_{+}>_{n} \tag{17}
\end{equation*}
$$

Then using (15)- (17) in (14), we get

$$
\begin{aligned}
\|\mathbf{e}\|_{L_{2}^{\Psi^{-1}}(\Omega)}^{2} & =\sum_{n=0}^{N-1}\left(\mathbf{e}_{t}, \Phi\right)_{n}+\sum_{n=0}^{N-1}(A \mathbf{e}, \Phi)_{n}-\sum_{n=0}^{N-1}<\left[\mathbf{w}_{h}\right], \Phi_{+}>_{n} \\
& =\sum_{n=0}^{N-1}\left(\left(\mathbf{w}-\mathbf{w}_{h}\right)_{t}+A\left(\mathbf{w}-\mathbf{w}_{h}\right), \Phi\right)_{n}-\sum_{n=0}^{N-1}<\left[\mathbf{w}_{h}\right], \Phi_{+}>_{n} \\
& =\sum_{n=0}^{N-1}\left(F-\mathbf{w}_{h, t}-A \mathbf{w}_{h}, \Phi\right)_{n}-\sum_{n=0}^{N-1}<\left[\mathbf{w}_{h}\right], \Phi_{+}>_{n} .
\end{aligned}
$$

Recalling (12) and using the Galerkin orthogonality (10), we obtain the final form of the error representation formula

$$
\begin{equation*}
\|\mathbf{e}\|_{L_{2}^{\Psi^{-1}}(\Omega)}^{2}=\sum_{n=0}^{N-1}\left(\mathbf{w}_{h, t}+A \mathbf{w}_{h}-F, \hat{\Phi}-\Phi\right)_{n}+\sum_{n=0}^{N-1}<\left[\mathbf{w}_{h}\right],(\hat{\Phi}-\Phi)_{+}>_{n} \equiv I+I I \tag{18}
\end{equation*}
$$

Where $\hat{\Phi} \in \mathbf{W}_{h}$ is an interpolate of $\Phi$. The idea is now to estimate $\hat{\Phi}-\Phi$ in terms of $\Psi^{-1} \mathbf{e}$ using a strong stability estimates for solution $\Phi$ of the dual problem (12).

### 3.2 A posteriori error estimates for the dual solution in $L_{2}\left(L_{2}\right)$

In this subsection for the interpolate $\hat{\Phi} \in \mathbf{U}_{h}$ in (18), we consider a certain space-time $L_{2}$-projection of $\Phi$ . For this purpose, we define the projections:

$$
P_{n}:\left[H_{0}^{1} \times L_{2}\right] \mapsto \mathbf{W}_{h}^{n},
$$

and the local time averages

$$
\pi_{n}:\left[L_{2}\left(S_{n}\right)\right]^{2} \mapsto \Pi_{0, n}=\left\{\mathbf{w} \in\left[L_{2}\left(S_{n}\right)\right]^{2}: \mathbf{w}(x, .) \text { isconstanton } I_{n}, x \in \Omega\right\}
$$

such that

$$
\begin{gathered}
\int_{\Omega}\left(P_{n} \Phi\right)^{T} \cdot \mathbf{w} d x=\int_{\Omega} \Phi^{T} \cdot \mathbf{w} d x, \quad \forall \mathbf{w} \in \mathbf{W}_{h}^{n}, \\
\left.\pi_{n} \mathbf{w}\right|_{S_{n}}=\frac{1}{k_{n}} \int_{I_{n}} \mathbf{w}(., t) d t, \quad \forall \mathbf{w} \in \Pi_{0, n} .
\end{gathered}
$$

Then, we define $\left.\hat{\Phi}\right|_{s_{n}} \in \mathbf{W}_{h}^{n}$ as

$$
\left.\hat{\Phi}\right|_{S_{n}}=P_{n} \pi_{n} \Phi=\pi_{n} P_{n} \Phi \in \mathbf{W}_{h}^{n}
$$

where $\Phi=\left.\Phi\right|_{S_{n}}$. Further, introducing $P$ and $\pi$ by the formulas

$$
\left.(P \Phi)\right|_{S_{n}}=P_{n}\left(\left.\Phi\right|_{S_{n}}\right),
$$

and

$$
\left.(\pi \Phi)\right|_{S_{n}}=\pi_{n}\left(\left.\Phi\right|_{S_{n}}\right)
$$

respectively, we can choose $\hat{\Phi} \in \mathbf{W}_{h}$ such that $\hat{\Phi}=P \pi \Phi=\pi P \Phi$.
Now, we define the residuals for the computed solution $\mathbf{w}_{h}$ by

$$
\begin{gathered}
R_{0}=\mathbf{w}_{h, t}+A \mathbf{w}_{h}-F, \\
R_{1}=\frac{\left(\mathbf{w}_{h,+}^{n}-\mathbf{w}_{h,-}^{n}\right)}{k_{n}}, \quad \text { on } S_{n}, \\
R_{2}=\frac{\left(P_{n}-l\right) \mathbf{w}_{h,-}^{n}}{k_{n}}, \quad \text { on } S_{n},
\end{gathered}
$$

where $I$ is the identity operator. Below, in our a posteriori approach, we will see how these residuals appear in a natural way.

To estimates $I$ and $I I$, we use stability estimates based on the following interpolation estimate for the projection operator $P$.

Lemma 3.1 There is a constant $C$ such that for a given residual $R \in L_{2}(\Omega)$,

$$
\begin{equation*}
\left|(R, \Phi-P \Phi)_{\Omega}\right| \leq C\left\|h^{2}(\mathrm{I}-P) R\right\|_{L_{2}^{\Psi^{-1}}(\Omega)}\|\Phi\|_{\dot{H}^{2, \Psi}(\Omega)}, \tag{19}
\end{equation*}
$$

where $\|\cdot\|_{\dot{H}^{2, \Psi}(\Omega)}$ is the $\Psi$-weighted seminorm.
We omit The proof since the arguments generalize the proof in the case of one spatial dimension presented in [10] (cf. also [14]).

Now, we prove the a posteriori error estimates by bounding the terms $I$ and $I I$ in the error representation formula (18). For this purpose, we introduce the stability factors (see [1] and [13]) associated with discretization in time and spatial variables and defined by the formulas

$$
\begin{equation*}
\gamma_{\mathbf{e}}^{t}=\frac{\left\|\Phi_{t}\right\|_{L_{2}^{\Psi}(\Omega)}}{\|\mathbf{e}\|_{L_{2}^{\Psi^{-1}(\Omega)}}} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{\mathrm{e}}^{x}=\frac{\|\Delta \Phi\|_{L_{2}^{\Psi}(\Omega)}}{\|\mathbf{e}\|_{L_{2}^{L^{-1}}(\Omega)}} \tag{21}
\end{equation*}
$$

respectively. We now combine (18), the interpolation estimate (19), the strong stability factors (20) and (21), we derive the $L_{2}\left(L_{2}\right)$ a posteriori error estimates for the finite element scheme (11).

Theorem 3.1 Let $\mathbf{w}$ be the solution to the continuous problem (1), and let $\mathbf{w}_{h}$ be its the finite element approximation given by (11). Then, the error $\mathbf{e}:=\mathbf{u}-\mathbf{u}_{h}$, satisfies the estimate

$$
\begin{align*}
\|\mathbf{e}\|_{L_{2}^{\psi^{-1}(\Omega)}} & \leq C \gamma_{\mathrm{e}}^{x}\left\|h^{2}(\mathrm{I}-P) R_{0}\right\|_{L_{2}^{\psi^{-1}(\Omega)}}+C \gamma_{\mathrm{e}}^{t}\left\|k_{n} R_{1}\right\|_{L_{2}^{\psi^{-1}(\Omega)}}  \tag{22}\\
& +\gamma_{\mathrm{e}}^{x}\left\|h^{2} R_{2}\right\|_{L_{2}^{\psi^{-1}(\Omega)}}+\gamma_{\mathrm{e}}^{t}\left\|k_{n} R_{2}\right\|_{L_{2}^{\psi^{-1}(\Omega)}} .
\end{align*}
$$

Proof. Using the above notation, from (19) we have

$$
\left.\|\mathbf{e}\|_{L_{2}^{\psi^{-1}(\Omega)}}^{2}=\sum_{n=0}^{N-1}\left(R_{0}, \hat{\Phi}-\Phi\right)_{n}+\sum_{n=0}^{N-1}<\left[\mathbf{w}_{h}\right],(\hat{\Phi}-\Phi)_{+}\right\rangle_{n}:=I+I I .
$$

We estimate $I$ and $I I$ separately. Writing

$$
\hat{\Phi}-\Phi=\hat{\Phi}-P \Phi+P \Phi-\Phi,
$$

and using $\hat{\Phi}_{n}=\pi_{n} P \Phi$, we find

$$
\begin{aligned}
I & =\sum_{n=0}^{N-1}\left(R_{0}, \hat{\Phi}_{n}-P \Phi+P \Phi-\Phi\right)_{n} \\
& =\sum_{n=0}^{N-1}\left(R_{0},\left(\pi_{n}-\mathrm{I}\right) P \Phi\right)_{n}+\sum_{n=0}^{N-1}\left(R_{0}, P \Phi-\Phi\right)_{n} \\
& \leq C\left\|h^{2}(\mathrm{I}-P) R_{0}\right\|_{L_{2}^{\Psi^{-1}(\Omega)}}\|\Phi\|_{\dot{H}^{2}, \Psi},
\end{aligned}
$$

where we used the fact that $R_{0}$ is constant in the time and, by the definition of the projections, the contribution of the first term in the first sum is zero. To estimate $I I$, we use (19) and

$$
\Phi_{+}^{n}(x)=\Phi(x, t)-\int_{t_{n}}^{t} \frac{\partial}{\partial \tau} \Phi(x, \tau) d \tau .
$$

Integrating over $I_{n}$, we find

$$
\begin{equation*}
k_{n} \Phi_{+}^{n}(x)=\int_{I_{n}} \Phi(x, t) d t-\int_{I_{n}} \int_{t_{n}}^{t} \Phi_{\tau}(x, \tau) d \tau d t, \tag{23}
\end{equation*}
$$

where $\Phi_{\tau}=\frac{\partial \Phi}{\partial \tau}$ and $\hat{\Phi}_{n}=\hat{\Phi}\left(., t_{n}\right)$. Now we rewrite $I I$ as follows:

$$
\begin{aligned}
I I & =\sum_{n=0}^{N-1}<k_{n} \frac{\left[\mathbf{w}_{h}\right]}{k_{n}},(\hat{\Phi}-\Phi)_{+}>_{n}=\sum_{n=0}^{N-1}<k_{n} \frac{\left[\mathbf{w}_{h}\right]}{k_{n}},\left(\hat{\Phi}_{n}-P \Phi+P \Phi-\Phi\right)_{+}>_{n} \\
& =\sum_{n=0}^{N-1}<k_{n} \frac{\left[\mathbf{w}_{h}\right]}{k_{n}},\left(\hat{\Phi}_{n}-P \Phi\right)_{+}>_{n}+\sum_{n=0}^{N-1}<k_{n} \frac{\left[\mathbf{w}_{h}\right]}{k_{n}},(P \Phi-\Phi)_{+}>_{n} \\
& :=I I_{1}+I I_{2} .
\end{aligned}
$$

To estimate $I I_{1}$, we use (23) and get

$$
\begin{aligned}
I I_{1} & =\sum_{n=0}^{N-1}\left\langle k_{n} R_{1},\left(\hat{\Phi}_{n}\right)_{+}-P \Phi_{+}\right\rangle_{n}=\sum_{n=0}^{N-1}\left\langle R_{1}, k_{n} \hat{\Phi}_{n}-P k_{n} \Phi_{+}\right\rangle_{n} \\
& =\sum_{n=0}^{N-1}\left\langle R_{1}, k_{n} \hat{\Phi}_{n}-\int_{I_{n}} P \Phi(., t) d t+\int_{I_{n}} \int_{t_{n}}^{t} P \Phi_{\tau}(., \tau) d \tau d t\right\rangle_{n} \\
& =\sum_{n=0}^{N-1} \int_{I_{n}} \int_{t_{n}}^{t}\left\langle R_{1}, P \Phi_{\tau}(., \tau)\right\rangle_{n} d \tau d t \\
& \leq\left\|k_{n} R_{1}\right\|_{L^{\Psi-1}\left(\Omega_{T}\right)}\left\|P \Phi_{t}\right\|_{L_{2}^{\Psi}\left(\Omega_{T}\right)} \\
& \leq\left\|k_{n} R_{1}\right\|_{L_{2}^{L^{\psi-1}\left(\Omega_{T}\right)}}\left\|\Phi_{t}\right\|_{L_{2}^{\Psi}\left(\Omega_{T}\right)},
\end{aligned}
$$

where $\Omega_{T}:=\Omega \times[0, T]$. As for the $I I_{2}$-terms we can write

$$
\begin{aligned}
I I_{2} & =\sum_{n=0}^{N-1}\left\langle k_{n} \frac{\left[\mathbf{w}_{h}\right]}{k_{n}},(P \Phi-\Phi)_{+}\right\rangle{ }_{n}=\sum_{n=0}^{N-1}\left\langle\frac{\mathbf{w}_{h,+}^{n}-\mathbf{w}_{h,-}^{n}}{k_{n}},\left(P_{n}-\mathrm{I}\right) k_{n} \Phi_{+}\right\rangle_{n} \\
& \left.=\sum_{n=0}^{N-1}<\frac{P_{n} \mathbf{w}_{h,-}^{n}-\mathbf{w}_{h,-}^{n}}{k_{n}},\left(P_{n}-\mathrm{I}\right)\left(\int_{I_{n}} \Phi(., t) d t-\int_{I_{n}} \int_{t_{n}}^{t} \Phi_{\tau}(., \tau) d \tau d t\right)\right\rangle_{n} \\
& \leq \sum_{n=0}^{N-1} \int_{I_{n}}\left\langle\frac{\left(P_{n}-I\right) \mathbf{w}_{h,-}^{n}}{k_{n}},\left(P_{n}-I\right) \Phi(., t)\right\rangle_{n} d t \\
& \left.-\sum_{n=0}^{N-1} \int_{I_{n}} \int_{t_{n}}^{t}<\frac{\left(P_{n}-I\right) \mathbf{w}_{h,-}^{n}}{k_{n}},\left(P_{n}-I\right) \Phi_{\tau}(., t) d \tau d t\right\rangle_{n} \\
& \leq\left\|k_{n} R_{2}\right\|_{L_{2}^{\Psi-1}\left(\Omega_{T}\right)}^{\|\Delta \Phi\|_{\dot{H}^{2, \Psi}\left(\Omega_{T}\right)}+\left\|k_{n} R_{2}\right\|_{L_{2}^{\Psi-1}\left(\Omega_{T}\right)}\left\|\Phi_{t}\right\|_{L_{2}^{\Psi}\left(\Omega_{T}\right)} .} .
\end{aligned}
$$

The final estimate is obtained by collecting the terms and using the definition of the stability factors (20) and (21).

### 3.3 A posteriori error estimates in $L_{\infty}\left(L_{2}\right)$

We derive a posteriori error bounds in the $L_{\infty}\left(L_{2}\right)$-norm for the scheme (11). For this purpose, we introduce the dual problem

$$
\left\{\begin{array}{cc}
L^{*} \Phi \equiv-\Phi_{t}+A^{T} \Phi=0, & \text { in } \Omega,  \tag{24}\\
\Phi(x, T)=E, & x \in \Omega
\end{array}\right.
$$

where $E$ satisfies the Poisson equation

$$
\begin{equation*}
-\Delta E=\mathbf{e}, \quad e(x)=w(x)-w_{h}(x) \quad x \in \Omega \tag{25}
\end{equation*}
$$

We introduce the energy norm

$$
\|\mathbf{e}\|_{L_{2}(\Omega)}=(\nabla E(T), \nabla E(T))_{\Omega}^{1 / 2}
$$

Using (25) and integrating by parts, we get

$$
\begin{aligned}
\|\mathbf{e}\|_{L_{2}(\Omega)}^{2} & =\left\|\nabla_{x} E\right\|_{L_{2}(\Omega)}^{2} \\
& =\left\langle\mathbf{e}_{-}, \Phi>_{N}+\sum_{n=0}^{N-1}\left(\mathbf{e}, L^{*} \Phi\right)_{n}=\left\langle\mathbf{e}_{-}, \Phi\right\rangle_{N}-\sum_{n=0}^{N-1}\left(\mathbf{e},-\Phi{ }_{t}+A^{T} \Phi\right)_{n}\right. \\
& =\left\langle\mathbf{e}_{-}, \Phi>_{N}-\sum_{n=0}^{N-1} \mathbf{e}^{T} .\left.\Phi\right|_{t_{n}} ^{t_{n+1}}+\sum_{n=0}^{N-1}\left(\mathbf{e}_{t}, \Phi\right)_{n}+\sum_{n=0}^{N-1}(A \mathbf{e}, \Phi)_{n}\right. \\
& =\sum_{n=0}^{N-1}\left(\mathbf{e}_{t}, \Phi\right)_{n}+\sum_{n=0}^{N-1}(A \mathbf{e}, \Phi)_{n}-\sum_{n=0}^{N-1}\left\langle\left[\mathbf{w}_{h}\right], \Phi_{+}\right\rangle_{n} \\
& =\sum_{n=0}^{N-1}\left(F-\mathbf{w}_{h, t}-A \mathbf{w}_{h}, \Phi\right)_{n}-\sum_{n=0}^{N-1}\left\langle\left[\mathbf{w}_{h}\right], \Phi_{+}>_{n}\right.
\end{aligned}
$$

Using the Galerkin orthogonality (4), we can subtract for $\hat{\Phi} \in \mathbf{W}_{h}$ from $\Phi$ on the right -hand side without changing the norm

$$
\begin{equation*}
\left.\|\mathbf{e}\|_{L_{2}(\Omega)}^{2}=\sum_{n=0}^{N-1}\left(\mathbf{w}_{h, t}+A \mathbf{w}_{h}-F, \hat{\Phi}-\Phi\right)_{n}+\sum_{n=0}^{N-1}<\left[\mathbf{w}_{h}\right],(\hat{\Phi}-\Phi)_{+}\right\rangle_{n} \tag{26}
\end{equation*}
$$

Here we again need to introduce the stability factors (cf. (20)-(21)), but this time in modified norms, adequate in the study of the full discrete (space-time discretization) problem in the $L_{\infty}$-norm:

$$
\begin{equation*}
\gamma_{E}^{t}=\frac{\left\|\Phi_{t}\right\|_{L_{1}\left(L_{2}(\Omega)\right)}}{\|\mathbf{e}\|_{L_{2}(\Omega)}} \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{E}^{x}=\frac{\|\Phi\|_{L_{1}\left(L_{2}(\Omega)\right)}}{\|\mathbf{e}\|_{L_{2}(\Omega)}} \tag{28}
\end{equation*}
$$

Using the interpolation estimates (19) and arguing in a similar way as in the proof of the Theorem 3.1, we get the following $L_{\infty}\left(L_{2}\right)$-estimate.

Theorem 3.2 Let $\mathbf{w}$ and $\mathbf{w}_{h}$ be the same as in Theorem 3.1. Then the error $\mathbf{e}:=w-w_{h}$ satisfies the estimate:

$$
\begin{align*}
\|\mathbf{e}\|_{L_{2}(\Omega)} & \leq C \gamma_{E}^{x}\left\|h^{2}(I-P) R_{0}\right\|_{L_{\infty}\left(L_{2}(\Omega)\right)}+C \gamma_{E}^{t}\left\|k_{n} R_{1}\right\|_{L_{\infty}\left(L_{2}(\Omega)\right)}  \tag{29}\\
& +\gamma_{E}^{x}\left\|h^{2} R_{2}\right\|_{L_{\infty}\left(L_{2}(\Omega)\right)}+\gamma_{E}^{t}\left\|k_{n} R_{2}\right\|_{L_{\infty}\left(L_{2}(\Omega)\right)},
\end{align*}
$$

where

$$
\begin{aligned}
\|\Phi\|_{L_{\infty}\left(L_{2}(\Omega)\right)} & =\sup _{0<t<T}\|\Phi(\cdot, t)\|_{L_{2}(\Omega)} \\
\|\Phi\|_{L_{1}\left(L_{2}(\Omega)\right)} & =\int_{0}^{T}\|\Phi(\cdot, t)\|_{L_{2}(\Omega)} d t
\end{aligned}
$$

The proof of this theorem is a modification of that of Theorem 3.1 and, therefore, is omitted. The only difference consists in the use of the Hölder inequality $\|f g\|_{1} \leq\|f\|_{p}\|g\|_{q}, 1 \leq p, q \leq \infty, 1 / p+1 / q=1 \quad(p=q=2$ in Theorem 3.1, whereas $p=1, q=\infty$ in Theorem 3.2).

## 4 Analytical strong stability estimates in $L_{2}\left(L_{2}\right)$

We need to estimate the strong stability factors used in the previous sections. Let us consider the a posteriori error estimate of the type (22) in Theorem 3.1 which is based on dual problem

$$
\left\{\begin{array}{cc}
L^{*} \Phi \equiv-\Phi_{t}+A^{T} \Phi=\Psi^{-1} \mathbf{e}, & \text { in } \Omega  \tag{30}\\
\Phi(x, T)=0, & x \in \Omega
\end{array}\right.
$$

We prove a strong stability estimate for dual problem (30).
Theorem 4.1 For a given positive weight function $\Psi(x, t)$ the solution $\Phi$ of the dual problem (30) satisfies the estimate

$$
\left\|\Psi^{1 / 2}\left(\Phi_{t}-A^{T} \Phi\right)\right\|_{\Omega}=\|\mathbf{e}\|_{L_{2}^{\Psi^{-1}}(\Omega)}
$$

Proof. We multiply in the equation (30) by $-\Psi\left(\Phi_{t}-A^{T} \Phi\right)$ and integrate over $\Omega$ to get

$$
\int_{\Omega} \Psi\left(\Phi_{t}-A^{T} \Phi\right)^{2} d x=-\int_{\Omega} \mathbf{e}\left(\Phi_{t}-A^{T} \Phi\right) d x \leq \frac{1}{2}\left\|\Psi^{-1 / 2} \mathbf{e}\right\|_{\Omega}^{2}+\frac{1}{2}\left\|\Psi^{1 / 2}\left(\Phi_{t}-A^{T} \Phi\right)\right\|_{\Omega}^{2}
$$

This yields

$$
\begin{equation*}
\left\|\Psi^{1 / 2}\left(\Phi_{t}-A^{T} \Phi\right)\right\|_{\Omega}^{2} \leq\left\|\Psi^{-1 / 2} \mathbf{e}\right\|_{\Omega}^{2} \tag{31}
\end{equation*}
$$

Similarly, multiplying the equation in (30) by $\mathbf{e}$ and integrating over $\Omega$, we get

$$
\begin{aligned}
\int_{\Omega} \mathbf{e}^{2} \Psi^{-1} d x & =\left\|\Psi^{-1 / 2} \mathbf{e}\right\|_{\Omega}^{2}=-\int_{\Omega} \Psi^{1 / 2}\left(\Phi_{t}-A^{T} \Phi\right)^{2} d x \\
& \leq \frac{1}{2}\left\|\Psi^{-1 / 2} \mathbf{e}\right\|_{\Omega}^{2}+\frac{1}{2}\left\|\Psi^{1 / 2}\left(\Phi_{t}-A^{T} \Phi\right)\right\|_{\Omega}^{2}
\end{aligned}
$$

which yields

$$
\begin{equation*}
\left\|\left\|\Psi^{-1 / 2} \mathbf{e}\right\|_{\Omega}^{2} \leq \mid \Psi^{1 / 2}\left(\Phi_{t}-A^{T} \Phi\right)\right\|_{\Omega}^{2} \tag{32}
\end{equation*}
$$

Combining (31) and (32), we complete the proof.
Theorem 4.2 If $\Psi(x, t)$ is a positive weight function such that

$$
\begin{equation*}
\Psi_{t}+A^{T} \Psi \geq-\Psi, \quad \text { in } \Omega \tag{33}
\end{equation*}
$$

then the solution $\Phi$ to the problem (30) satisfies the estimate

$$
\left\|\Psi^{1 / 2} \Phi\right\|_{\Omega} \leq C_{T}\|e\|_{L_{2}^{\Psi^{-1}(\Omega)}}, \quad C_{T}=\sqrt{T} e^{T}
$$

Proof. Multiplying the equation in (30) by $\Psi \Phi$ and integrating over $\Omega$, we get the equality

$$
-\left(\Phi_{t}, \Psi \Phi(t)\right)+\left(A^{T} \Phi, \Psi \Phi(t)\right)=(\mathbf{e}, \Phi(t))
$$

which can be written as

$$
-\frac{1}{2} \frac{d}{d t}\left\|\Psi^{1 / 2} \Phi(t)\right\|^{2}+\frac{1}{2}\left(\Psi_{t}, \Phi^{2}(t)\right)+\left(A^{T} \Phi, \Psi \Phi(t)\right)=(\mathbf{e}, \Phi(t))
$$

Integrating by parts in spatial variables and then using (30) together with Cauchy-Schwarz inequality, we get
$-\frac{1}{2} \frac{d}{d t}\left\|\Psi^{1 / 2} \Phi(t)\right\|^{2}+\frac{1}{2}\left(\Psi_{t}+A^{T} \Psi, \Phi^{2}(t)\right) \leq\left\|\Psi^{-1 / 2} \mathbf{e}\right\|\left\|\mid \Psi^{1 / 2} \Phi\right\| \leq \frac{1}{2}\left\|\Psi^{-1 / 2} \mathbf{e}\right\|^{2}+\frac{1}{2}\left\|\Psi^{1 / 2} \Phi\right\|^{2}$.
Using (33), we find

$$
-\frac{1}{2} \frac{d}{d t}\left\|\Psi^{1 / 2} \Phi(t)\right\|^{2}-\frac{1}{2}\left(\Psi, \Phi^{2}(t)\right) \leq \frac{1}{2}\left\|\Psi^{-1 / 2} \mathbf{e}\right\|^{2}+\frac{1}{2}\left\|\Psi^{1 / 2} \Phi\right\|^{2}
$$

Integrating in the time variable over $(t, T)$ and using the equality $\Psi(., T)=0$, we get

$$
\left\|\Psi^{1 / 2} \Phi(t)\right\|^{2} \leq\left\|\Psi^{-1 / 2} \mathbf{e}\right\|_{\Omega}^{2}+2 \int_{t}^{T}\left\|\Psi^{1 / 2} \Phi(t)\right\|^{2} d s
$$

By the Gronwall inequality, we arrive at the desired result

$$
\left\|\Psi^{1 / 2} \Phi(t)\right\|^{2} \leq \mathbf{e}^{2 T}\left\|\Psi^{-1 / 2} \mathbf{e}\right\|_{\Omega}^{2}
$$

The theorem is proved.
The proof for the analytical strong stability estimates in $L_{\infty}\left(L_{2}\right)$ is similar to the $L_{2}\left(L_{2}\right)$ case.

## 5. Conclusion

To this end, a spatial linear second order hyperbolic initial-boundary value problem is investigated. We use streamline diffusion method for generalizing telgraph equation and obtain a
posteriori error estimates. A posteriori error estimate is a very powerful mathematical tool in this problem by SD method. We try to obtain optimal bounds and the spatial numerical results reminds challenge that deserves special attention and will be consideration elsewhere.

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