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Taylor Polynomial solution of Nonlinear Mixed Volterra-Fredholm-Hammerstein Integral Equations

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ABSTRACT

This paper presents a computational technique for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equations. The method is based on the Taylor polynomials. Nonlinear integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed. A considerable advantage of the method is that the solution is expressed as a truncated Taylor series.

Keywords: Mixed Volterra – Fredholm – Hamerstein , integral equations, Taylor -polynomials.

1. INTRODUCTION

There is considerable literature that discussed approximating the solution of linear and nonlinear Hammerstein integral equatins [1,2,3,5,6,7,8,9,13,14]. A Taylor expansion approach for solving integral equations has been presented by kanwal and liu [4] and then this has been extended by Sezar to Volterra integral equations [10] and to differential equations [11]. The technique is based on, first, differentiating both sides of the unknown function in the resulting equation and later, transforming to a matrix equation.

In this study, the basic of the previous works are developed and applied to the nonlinear Volterra-Fredholm-Hammerstein integral equation

$$y(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)\left[y(t)\right]^p + \int_a^b K_2(x,t)\left[y(t)\right]^q dt, \qquad (1)$$

where p is positive integer and $q=1$, $f(x), K_1(x,t)$ and $K_2(x,t)$ are functions
Having n th derivatives on an interval $a \le x,t \le b,$ and a,b,λ_1,λ_2 are constants;
and the solution is expressed in the form

$$y(x) = \sum_{n=0}^{N} \frac{1}{n!} y^{(n)}(c)(x-c)^n, \qquad a \le x, c \le b, \qquad (2)$$

Which is a Taylor polynomial of degree N at $x=c,$ where $y^{(n)}(c), n = 0,1,\ldots,N$ are coefficients to be determined.

2. The method of solution

To obtain the solution of equation (1) in the form of expression (2) we first differentiate it n times with respect to x:

$$y^{(n)}(x) = f^{(n)}(x) + \lambda_1 V^{(n)}(x) + \lambda_2 F^{(n)}(x), \qquad (3)$$

Where

$$V^{(n)}(x) = \frac{d^n}{dx^n} \int_a^x K_1(x,t)\left[y(t)\right]^p dt, \qquad (4)$$

$$F^{(n)}(x) = \int_a^b \frac{\partial^n K_2(x,t)}{\partial x^n} y(t) dt. \qquad (5)$$

We now consider eq. (4). Substituting the expression

$$Y(t) = \left[y(t)\right]^p$$

in Eq. (4), we obtain

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$$V^{(n)}(x)=\frac{d^n}{dx^n}\int_a^x K_1(X,T)Y(t)dt. \qquad (6)$$

For $n=0$

$$V^{(0)}(x)=V(x)=\int_a^x K_1(x,t)Y(t)dt.$$

By applying successively n times the Leibnitz's rule (dealing with differentiation of integrals) to the integral $V(x)$, we have, for $n\geq 1$

$$V^{(n)}(x)=\sum_{i=0}^{n-1}\left[h_i(x)Y(x)\right]^{(n-i-1)}+\int_a^x \frac{\partial^{(n)}K_1(x,t)}{\partial x^n}Y(t)dt, \qquad (7)$$

where

$$h_i(x)=\frac{\partial^{(i)}K_1(x,t)}{\partial x^i}\downarrow_{t=x}. \qquad (8)$$

Form the Leibnitz's rule (dealing with differentiation of products of functions), we evaluate $\left[h_i(x)Y(x)\right]^{(n-i-1)}$ and substitute it in Eq. (7). Thus, Eq. (6) becomes

$$V^{(n)}(x)=\sum_{m=0}^{n-1}\sum_{i=0}^{n-m-1}\binom{n-i-1}{m}h_i^{(n-m-i-1)}(x)Y^{(m)}(x)+\int_a^x \frac{\partial^{(n)}K_1(x,t)}{\partial x^n}Y(t)dt. \qquad (9)$$

Where

$$c_m=\binom{n-i-1}{m}=\frac{(n-i-1)!}{m!(n-i-m-1)!}$$

Thus, Eq. (9) becomes

$$V^{(n)}(x)=\sum_{i=0}^{n-1}\sum_{m=0}^{n-i-1}c_m h_i^{(n-m-i-1)}(x)Y^{(m)}(x)+\int_a^x \frac{\partial^{(n)}K_1(x,t)}{\partial x^n}Y(t)dt.$$

Note that in Eq. (9)

$$\sum_{m=0}^{n-1}\sum_{i=0}^{n-m-1}(\cdots)=\sum_{i=0}^{n-1}\sum_{m=0}^{n-i-1}(\cdots).$$

First, we put $x=c$ in relation (3), thereby in expressions (5) and (9), and then substitute the Taylor expansions of $y(t)$ and $Y(t)$ at $t=c$, i.e.

$$Z(t)=\sum_{m=0}^{\infty}\frac{1}{m!}y^{(m)}(c)(t-c)^m, \qquad Y(t)=\sum_{m=0}^{\infty}\frac{1}{m!}Y^{(m)}(c)(t-c)^m$$

Therefore we have:

$$y^{(n)}(c)=f^{(n)}(c)+\lambda_1\sum_{m=0}^{n-1}\sum_{i=0}^{n-m-1}c_m h_i^{(n-i-m-1)}(c)Y^{(m)}(c)$$

$$+\lambda_1\int_a^c \frac{\partial^n K_1(x,t)}{\partial x^n}\downarrow_{x=c}\left[\sum_{m=0}^{\infty}\frac{1}{m!}Y^{(m)}(c)(t-c)^m\right]dt$$

$$+\lambda_2\int_a^b \frac{\partial^n K_2(x,t)}{\partial x^n}\downarrow_{x=c}\left[\sum_{m=0}^{\infty}\frac{1}{m!}Z^{(m)}(c)(t-c)^m\right]dt$$

Or briefly

$$y^{(n)}(c)=f^{(n)}(c)+\lambda_1\left\{\sum_{m=0}^{n-1}H_{nm}Y^{(m)}(c)+\sum_{m=0}^{\infty}T_{nm}Y^{(m)}(c)\right\}+\lambda_2\sum_{m=0}^{\infty}K_{nm}Z^{(m)}(c) \qquad (10)$$

In other words,

$$y^{(n)}(c)=f^{(n)}(c)+\lambda_1\left\{\sum_{m=0}^{n-1}(H_{nm}+T_{nm})Y^{(m)}(c)+\sum_{m=n}^{N}T_{nm}Y^{(m)}(c)\right\}+\lambda_2\sum_{m=0}^{N}K_{nm}Z^{(m)}(c)$$

Where for $n=0$

$$\sum_{m=0}^{n-1}(H_{nm}+T_{nm})Y^{(m)}(c)=0, \qquad \sum_{m=0}^{n-1}K_{nm}Z^{(m)}(c)=0$$

For $n=1,2,\ldots;\quad n=1,2,\ldots;\qquad m=0,1,\ldots,n-1(n>m)$

$$H_{nm}=\sum_{i=0}^{n-m-1}\binom{n-i-1}{m}h_i^{(n-m-i-1)}(c) \qquad (11)$$

For $n\le m$

$$H_{nm}=0$$

And for $n,m=0,1,2$

$$T_{nm}=\frac{1}{m!}\int_a^c\frac{\partial^n K(x,t)}{\partial x^n}\downarrow_{x=c}(t-c)^m\,dt, \qquad (12)$$

$$K_{nm}=\frac{1}{m!}\int_a^b\frac{\partial K_2(x,t)}{\partial x^n}\downarrow_{x=c}(t-c)^m\,dt. \qquad (13)$$

The quantities $Y^{(m)}(c)(m=0,1,2,\ldots)$ in Eq. (10) can be found from the permutation relation

$$Y^{(m)}(c)=\sum_{t_1+t_2+\ldots+t_p=m}c_p y^{(m_2+t_1)}(c).y^{(m_2+t_2)}(c)\cdots y^{(m_2+t_p)}(c) \qquad (14)$$

$$Z^{(m)}(c)=\sum_{t_1+t_2+\ldots+t_p=m}c_q y^{(m_3+t_1)}(c).y^{(m_3+t_2)}(c)\cdots y^{(m_3+t_q)}(c)$$

$$c_p=\frac{m!}{t_1!t_2!\cdots t_p!}, \qquad c_q=\frac{m!}{t_1!t_2!\cdots t_q!}$$

If we take $n,m=0,1,2,\cdots,N$, then Eq. (10) becomes

$$y^{(0)}(c)=f^{(0)}(c)+\lambda_1\sum_{m=0}^{N}T_{0m}Y^{(m)}(c)+\lambda_2\sum_{m=0}^{N}K_{0m}Z^{(m)}(c),$$

$$n=1,2,\cdots;m=0,1,\cdots,N\Rightarrow y^{(n)}(c)=f^{(n)}(c)+\lambda_1\left\{\sum_{m=0}^{n-1}(H_{nm}+T_{nm})Y^{(m)}(c)+\sum_{m=n}^{N}T_{nm}Y^{(m)}(c)\right\}$$

$$+\lambda_2\sum_{m=0}^{N}K_{nm}Z^{(m)}(c) \qquad (15)$$

Which is a algebraic system of $N+1$ nonlinear equations for the $N+1$ unknowns these can be solved numerically by standard methods.

The system (15) can be put in a matrix form a matrix form as

$$Y-\lambda_1 TY^*-\lambda_2 KZ^*=F$$

where Y,F,K,T and Y^*,Z^* are matrices defined by

$$Y=\begin{bmatrix} y^{(0)}(c)\\ y^{(1)}(c)\\ y^{(2)}(c)\\ \vdots\\ y^{(N)} \end{bmatrix}, \qquad F=\begin{bmatrix} f^{(0)}(c)\\ f^{(1)}(c)\\ f^{(2)}(c)\\ \vdots\\ f^{(N)}(c) \end{bmatrix},$$

$$K=\begin{bmatrix} K_{00} & K_{01} & K_{02} & \cdots & K_{0N}\\ K_{10} & K_{11} & K_{12} & \cdots & K_{1N}\\ K_{20} & K_{21} & K_{22} & \cdots & K_{2N}\\ \vdots & \vdots & \vdots & & \vdots\\ K_{N0} & K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix},$$

$$T=\begin{bmatrix} T_{00} & T_{01} & T_{02} & \cdots & T_{0N}\\ (H_{10}+T_{10}) & T_{11} & T_{12} & \cdots & T_{1N}\\ (H_{20}+T_{20}) & (H_{21}+T_{21}) & T_{22} & \cdots & T_{2N}\\ \vdots & \vdots & \vdots & & \vdots\\ (H_{N0}+T_{N0}) & (H_{N1}+T_{N1}) & (H_{N2}+T_{N2}) & & T_{NN} \end{bmatrix},$$

And

$$Y^{*}=\left[Y^{(0)}(c)\quad Y^{(1)}(c)\quad \cdots\quad Y^{(N)}(c)\right]^{T},$$

$$Z^{*}=\left[Z^{(0)}(c)\quad Z^{(1)}(c)\quad \cdots\quad Z^{(N)}(c)\right]^{T}.$$

We can easily check the accuracy of this solution as follows. Since the truncated Taylor series or the corresponding polynomial expansion is an approximate solution of equation (1), when the solution $y(x)$ is substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for $x=\bar{x}\in[a,b]$

$$D(\bar{x})=\left|y(\bar{x})-f(\bar{x})-\lambda_1 V(\bar{x})-\lambda_2 F(\bar{x})\right|\cong 0, \qquad (17)$$

where

$$F(\bar{x})=\int_a^b K_2(\bar{x},t)Z(t)dt, \qquad V(\bar{x})=\int_a^{\bar{x}} K_1(\bar{x},t)Y(t)dt$$

or

$$D(\bar{x})\le 10^{-k},\ k\in z^{+}$$

If max $10^{-k}=10^{-k_r}$ (k_r is any positive integer) is prescribed, then the truncation limit N is increased until the difference $D(\bar{x})$ at each of the points becomes smaller than the prescribed 10^{-k_r} [12].

3. Illustrations

The method of this study is useful in finding the solutions of nonlinear Vorterra-Frerholm-Hammerstein integral equations in terms of Taylor polynomials. We illustrate it by the following examples.

4. Numerical examples

The method of this study is useful in finding the solutions of nonlinear Volterra- Fredholm- Hammerstein integral equations in terms of Taylor polynomials. We illustrate it by the following examples.

Example 1. Let us first consider the nonlinear Volterra-Fredholm-Hammerstein integral equation

$$y(x)=-\frac{1}{30}x^6+\frac{1}{3}x^4-x^2+\frac{5}{3}x-\frac{5}{4}+\int_0^x (x-t)[y(t)]^2dt+\int_0^1 (x+t)y(t)dt, \quad 0\le x,t\le 1$$

And approximate the solution $y(x)$ by the Taylor polynomial

$$y(x)=\sum_{n=0}^{5}\frac{1}{n!}y^{(n)}(0)x^n$$

So that $N=5,\ a=0, b=1, c=0, \lambda_1=1, \lambda_2=1,$

$$f(x)=-\frac{1}{30}x^6+\frac{1}{3}x^4-x^2+\frac{5}{3}x-\frac{5}{4},\quad K_1(x,t)=x-t,\quad K_2(x,t)=x+t.$$

First, let us find the coefficients H_{nm} from (8) and (11), the coefficients T_{nm} from (12), and the coefficients $k_{nm}(n,m=0,1,\dots,5)$ from (13), and then we get the derivation values of the function $f(x)$ at $x=0$ as

$$f^{(0)}(0)=-\frac{5}{4},\ f^{(1)}(0)=-\frac{5}{3},\ f^{(2)}(0)=-2,\ f^{(3)}(0)=0,\ f^{(4)}(0)=8,\ f^{(5)}(0)=0.$$

Then, for $N=5$, the matrix equation (17)

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} & -\frac{1}{8} & -\frac{1}{30} & -\frac{1}{144} & -\frac{1}{840} \\ -1 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{24} & -\frac{1}{120} & -\frac{1}{720} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^{(0)}(0) \\ y^{(1)}(0) \\ y^{(2)}(0) \\ y^{(3)}(0) \\ y^{(4)}(0) \\ y^{(5)}(0) \end{bmatrix}$$

$$-\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y^{(0)}(0) \\ Y^{(1)}(0) \\ Y^{(2)}(0) \\ Y^{(3)}(0) \\ Y^{(4)}(0) \\ Y^{(5)}(0) \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} \\ \frac{5}{3} \\ -2 \\ 0 \\ 8 \\ 0 \end{bmatrix}$$

From the obtained equation system, the coefficients are found as

$$y^{(0)}(0)=-2, y^{(1)}(0)=0, y^{(2)}(0)=2, y^{(3)}(0)=0, y^{(4)}(0)=0, y^{(5)}(0)=0.$$

Substituting these coefficients in (2) we have the solution $y(x)=x^2-2..$

Example 2. Let us now study the integral equation

$$y(x)=-\frac{15}{56}x^8+\frac{13}{14}x^7-\frac{11}{10}x^6+\frac{9}{20}x^5+x^2-x,\qquad K_1(x,t)=x+t. \tag{18}$$

And approximate the solution $y(x)$ by a Taylor polynomial of fifth degree, so that $c=0, N=5, a=0, \lambda_1=1,$

First, we find the coefficients H_{nm} from (8) and (11) as

$H_{10}=0$

$H_{20}=3 \quad H_{21}=0$

$H_{30}=0 \quad H_{31}=5 \quad H_{32}=0$

$H_{40}=0 \quad H_{41}=0 \quad H_{42}=7 \quad H_{43}=0$

$H_{50}=0 \quad H_{51}=0 \quad H_{52}=0 \quad H_{53}=9 \quad H_{54}=0$

And then we obtain the derivation value of $f(x)$ function at $x=0$ as

$$f^{(0)}(0)=0,\ f^{(1)}(0)=-1,\ f^{(2)}(0)=2,\ f^{(3)}(0)=0,\ f^{(4)}(0)=0,\ f^{(5)}(0)=54.$$

Then, these coefficients are obtained as

$y^{(0)}(0)=0 \quad y^{(1)}(0)=-1 \quad y^{(2)}(0)=2 \quad y^{(3)}(0)=0 \quad y^{(4)}(0)=0$
$y^{(5)}(0)=0$
By means of (14) system. Thus, substituting these coefficients in (2), we get the solution of equation (18) as $y(x)=x^2-x.$

5. CONCLUSIONS

Nonlinear integral equations are usually to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed. A considerable advantage of the method is that the solution is expressed as a truncated Taylor series and thereby a Taylor polynomial at $x=c$ .furthermore, after calculation of the series coefficients, the solution $y(x)$ can be easily evaluated for arbitrary values of x at low computation effort.

If the function $f(x), K_1(x,t), K(x,t)$ are function having n th derivatives on the interval $a \le x, t \le b,$ then we can approach the solution $y(x)$ by the Taylor polynomial

$$y(x)=\sum_{n=0}^{N}\frac{1}{n!}y^{(n)}(c)(x-c)^n$$

About $x=c;$ otherwise, the method cannot be used.

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