

Numerical Solution of Tenth-Order Boundary-Value Problems in Off Step Points

Karim Farajeyan¹ and Nader Rafati Maleki²

¹Department of Mathematics, Bonab Branch, Islamic Azad University, Bonab, Iran

²Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran

ABSTRACT

Non-polynomial spline in off step points is used to solve special tenth order linear boundary value problems. Associated boundary formulas are developed. Truncation errors are given. Three examples are considered for the numerical illustration. However, it is observed that our approach produce better numerical solutions in the sense that $\max |e_i|$ is minimum.

KEYWORDS: Tenth -order boundary-value problem; boundary formulae; Non-polynomial spline.

1 INTRODUCTION

We consider tenth -order boundary-value problem of type

$$y^{(10)}(x) + f(x)y(x) = g(x), \quad x \in [a, b] \quad (1)$$

With boundary conditions

$$\begin{aligned} y(a) &= \alpha_0, \quad y^{(1)}(a) = \alpha_1, \quad y^{(2)}(a) = \alpha_2, \quad y^{(3)}(a) = \alpha_3, \quad y^{(4)}(a) = \alpha_4 \\ y(b) &= \beta_0, \quad y^{(1)}(b) = \beta_1, \quad y^{(2)}(b) = \beta_2, \quad y^{(3)}(b) = \beta_3, \quad y^{(4)}(b) = \beta_4 \end{aligned} \quad (2)$$

where α_i, β_i for $i = 0, 1, 2, 3, 4$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$.

The solution of tenth order boundary value problems are not very much found in the analysis literature. When instability sets in an ordinary convection, it is modeled by tenth-order boundary value problem[1]. Twizell et al.[2] developed numerical methods for 8th-, 10th-, and 12th-order eigenvalue problems arising in thermal instability. Siddiqi and Twizell[3] presented the solution of 10th-order boundary value problem using 10th degree spline. Siddiqi and Akram[4] developed the solution of 10th-order boundary value problems using non-polynomial spline. Siddiqi and Akram[5] presented the solution of 10th-order boundary value problem by using eleventh degree spline. Rashidinia et al.[6] developed numerical methods for 8th-order boundary value problem using non-polynomial spline.

Following the spline functions proposed in this paper have the form $T_{11} = \text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, \cos(kx), \sin(kx)\}$ where k is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \leq x \leq x_{i+1}$ we have

$$\begin{aligned} &\text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, \cos(|k|x), \sin(|k|x)\}, \text{ or} \\ &\text{span}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}\}, (\text{When } k \rightarrow 0) \end{aligned}$$

In this paper, in Section 2, the new Non-polynomial spline methods are developed for solving equation (1) along with boundary condition (2). Development of the boundary formulas are considered in Section 3. In Section 4, non-polynomial spline solution of (1)and(2) is determined and in Section 5 numerical experiment, discussion are given.

2 NUMERICAL METHODS

To develop the spline approximation to the tenth -order boundary-value problem (1)-(2), the interval $[a,b]$ is divided in to n equal subintervals using the grid $x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h$,

***Corresponding Author:** Karim Farajeyan, Department of Mathematics, Bonab Branch, Islamic Azad University, Bonab, Iran. Email:karim_faraj@yahoo.com

$i = 1, 2, 3, \dots, n$ where $h = \frac{b-a}{n}$. We Consider the following Non-polynomial spline $S_i(x)$ on each Subinterval $[x_{\frac{i-1}{2}}, x_{\frac{i+1}{2}}]$, $i = 0, 1, 2, \dots, n-1$, $x_0 = a$, $x_n = b$,

$$\begin{aligned} S_i(x) = & a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i (x - x_i)^9 + d_i (x - x_i)^8 + e_i (x - x_i)^7 + \\ & f_i (x - x_i)^6 + g^*_i (x - x_i)^5 + o_i (x - x_i)^4 + p_i (x - x_i)^3 + q^*_i (x - x_i)^2 + r_i (x - x_i) + u_i \end{aligned} \quad (3)$$

where $a_i, b_i, c_i, d_i, e_i, f_i, g^*_i, o_i, p_i, q^*_i, r_i$ and u_i are real finite constants and k is free parameter. The spline is defined in terms of its 1td, 2th, 3th, 4th and 10th derivatives and we denote these values at knots as:

$$\begin{aligned} S_i(x_{\frac{i-1}{2}}) = & y_{\frac{i-1}{2}}, S'_i(x_{\frac{i-1}{2}}) = m_{\frac{i-1}{2}}, S''_i(x_{\frac{i-1}{2}}) = M_{\frac{i-1}{2}}, S'''_i(x_{\frac{i-1}{2}}) = z_{\frac{i-1}{2}}, S^{(4)}_i(x_{\frac{i-1}{2}}) = V_{\frac{i-1}{2}} \\ S_i(x_{\frac{i+1}{2}}) = & y_{\frac{i+1}{2}}, S'_i(x_{\frac{i+1}{2}}) = m_{\frac{i+1}{2}}, S''_i(x_{\frac{i+1}{2}}) = M_{\frac{i+1}{2}}, S'''_i(x_{\frac{i+1}{2}}) = z_{\frac{i+1}{2}}, S^{(4)}_i(x_{\frac{i+1}{2}}) = V_{\frac{i+1}{2}} \\ S_i^{(10)}(x_{\frac{i-1}{2}}) = & L_{\frac{i-1}{2}}, S_i^{(10)}(x_{\frac{i+1}{2}}) = L_{\frac{i+1}{2}} \\ \text{For } i = & 0, 1, 2, \dots, n-1. \end{aligned} \quad (4)$$

Algebraic manipulation yields the following expressions, where by $\theta = kh$ and $i = 0, 1, 2, \dots, n-1$:

$$\begin{aligned} a_i = & \frac{1}{2k^{10}} (\sec(\frac{\theta}{2})(L_{\frac{i-1}{2}} + L_{\frac{i+1}{2}})), b_i = \frac{1}{2k^{10}} (cec(\frac{\theta}{2})(L_{\frac{i-1}{2}} - L_{\frac{i+1}{2}})), \\ c_i = & \frac{-h}{24\theta^{10}} ((1680 - 180\theta^2 + \theta^4 - 20\theta(42 - \theta^2)\cot(\frac{\theta}{2})(L_{\frac{i-1}{2}} - L_{\frac{i+1}{2}}) + k^{10}(840h(m_{\frac{i-1}{2}} + \\ & 180h^2(M_{\frac{i-1}{2}} - M_{\frac{i+1}{2}}) + h^4(V_{\frac{i-1}{2}} - V_{\frac{i+1}{2}}) + 1680(V_{\frac{i-1}{2}} - V_{\frac{i+1}{2}}) + 20h^3(z_{\frac{i-1}{2}} + z_{\frac{i+1}{2}})), \\ d_i = & \frac{h^2}{48\theta^9} ((\theta(-60 + \theta^2) + 12(10 - \theta^2)\tan(\frac{\theta}{2})(L_{\frac{i-1}{2}} + L_{\frac{i+1}{2}}) + k^9(120(m_{\frac{i-1}{2}} + m_{\frac{i+1}{2}})) + \\ & 60h(M_{\frac{i-1}{2}} + M_{\frac{i+1}{2}}) + h^3(V_{\frac{i-1}{2}} + V_{\frac{i+1}{2}}) + 12h^2(z_{\frac{i-1}{2}} + z_{\frac{i+1}{2}})), \\ e_i = & \frac{h^2}{24\theta^{10}} ((2160 - 228\theta^2 + \theta^4 - 24\theta(45 - \theta^2)\cot(\frac{\theta}{2})(L_{\frac{i-1}{2}} - L_{\frac{i+1}{2}}) + k^{10}(1080h(m_{\frac{i-1}{2}} + \\ & m_{\frac{i+1}{2}})) + 228h^2(M_{\frac{i-1}{2}} - M_{\frac{i+1}{2}}) + h^4(V_{\frac{i-1}{2}} - V_{\frac{i+1}{2}}) + 24h^3(z_{\frac{i-1}{2}} + z_{\frac{i+1}{2}}) + 2160(y_{\frac{i-1}{2}} - y_{\frac{i+1}{2}})), \\ f_i = & \frac{h^4}{48\theta^9} ((84\theta - \theta^3 + 8(-21 + 2\theta^2)\tan(\frac{\theta}{2})(L_{\frac{i-1}{2}} + L_{\frac{i+1}{2}}) + (-k^9(168(m_{\frac{i-1}{2}} + m_{\frac{i+1}{2}})) + \\ & 84h(M_{\frac{i-1}{2}} + M_{\frac{i+1}{2}}) + h^3(V_{\frac{i-1}{2}} + V_{\frac{i+1}{2}}) + 16h^2(z_{\frac{i-1}{2}} - z_{\frac{i+1}{2}})), \\ g^*_i = & \frac{h^5}{64\theta^{10}} ((3024 - 308\theta^2 + \theta^4 - 28\theta(54 + \theta^2)\cot(\frac{\theta}{2})(L_{\frac{i-1}{2}} - L_{\frac{i+1}{2}}) + k^{10}(1512h(m_{\frac{i-1}{2}} + \\ & m_{\frac{i+1}{2}})) + 308h^2(M_{\frac{i-1}{2}} - M_{\frac{i+1}{2}}) + h^4(V_{\frac{i-1}{2}} - V_{\frac{i+1}{2}}) + 28h^3(z_{\frac{i-1}{2}} + z_{\frac{i+1}{2}}) + 3024(y_{\frac{i-1}{2}} - y_{\frac{i+1}{2}})), \end{aligned}$$

$$\begin{aligned}
o_i &= \frac{h^6}{128\theta^9} (\theta(-140 + \theta^2) - 20(-14 + \theta^2)) \tan(\frac{\theta}{2}) (L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + (k^9 (280(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}})) + \\
&\quad 140h(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + h^3(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + 20h^2(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}})), \\
p_i &= \frac{h^7}{3840\theta^{10}} ((5040 - 420\theta^2 + \theta^4 - 8\theta(315 + 4\theta^2)) \cot(\frac{\theta}{2}) (L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}}) + k^{10} (2520h(m_{i-\frac{1}{2}} + \\
&\quad m_{i+\frac{1}{2}})) + 420h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + 32h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}) + 5040(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}})), \\
q_i^* &= \frac{h^8}{768\theta^9} ((228\theta - \theta^3 - 24(35 - \theta^2)) \tan(\frac{\theta}{2}) (L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + (-k^9 (840(m_{i-\frac{1}{2}} - m_{i+\frac{1}{2}})) + \\
&\quad 228h(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + h^3(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + 24h^2(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}})), \\
r_i &= \frac{-h^9}{6144\theta^{10}} ((15120 - 564\theta^2 + \theta^4 + 12\theta(-374 + 3\theta^2)) \cot(\frac{\theta}{2}) (L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}}) + k^{10} (4488h(m_{i-\frac{1}{2}} + \\
&\quad m_{i+\frac{1}{2}})) + 564h^2(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + h^4(V_{i-\frac{1}{2}} - V_{i+\frac{1}{2}}) + 36h^3(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}}) + 15120(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}})), \\
u_i &= \frac{h^{10}}{12288\theta^9} ((6144 - 384\theta^2 + \theta^4 + 4\theta(558 - 7\theta^2)) \tan(\frac{\theta}{2}) (L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + (k^{10} (2232h(m_{i-\frac{1}{2}} - \\
&\quad m_{i+\frac{1}{2}})) + 348h^2(M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}) + h^4(V_{i-\frac{1}{2}} + V_{i+\frac{1}{2}}) + 28h^3(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}}) + 6144(y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}})),
\end{aligned}$$

Assuming $y(x)$ to be the exact solution of the boundary value problem (1) and y_i be an approximation to $y(x_i)$ using the continuity conditions ($S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i)$ where $\mu = 5, 6, 7, 8$ and 9), we obtain the following spline relations:

$$\begin{aligned}
(y_{i-\frac{11}{2}} + y_{i+\frac{9}{2}}) - 10(y_{i-\frac{9}{2}} + y_{i+\frac{7}{2}}) + 45(y_{i-\frac{7}{2}} + y_{i+\frac{5}{2}}) - 120(y_{i-\frac{5}{2}} + y_{i+\frac{3}{2}}) + 210(y_{i-\frac{3}{2}} + y_{i+\frac{1}{2}}) - \\
252y_{i-\frac{1}{2}} = h^{10} [\alpha(L_{i-\frac{11}{2}} + L_{i+\frac{9}{2}}) + \beta(L_{i-\frac{9}{2}} + L_{i+\frac{7}{2}}) + \gamma(L_{i-\frac{7}{2}} + L_{i+\frac{5}{2}}) + \delta(L_{i-\frac{5}{2}} + L_{i+\frac{3}{2}}) + \\
\eta(L_{i-\frac{3}{2}} + L_{i+\frac{1}{2}}) + \lambda L_{i-\frac{1}{2}}] , \quad i = 6, 7, \dots, n-6.
\end{aligned} \tag{5}$$

Where

$$\begin{aligned}
\alpha &= \frac{\csc \theta}{\theta} \left(\frac{1}{362880} + \frac{1}{\theta^8} - \frac{1}{6\theta^6} + \frac{1}{120\theta^4} - \frac{1}{5040\theta^2} - \frac{\sin \theta}{\theta^9} \right), \\
\beta &= \frac{\csc \theta}{\theta} \left(\frac{251 - \cos \theta}{181440} - \frac{59 - \cos \theta}{2520\theta^2} + \frac{11 - \cos \theta}{60\theta^4} + \frac{1 + \cos \theta}{3\theta^6} - \frac{8 + 2\cos \theta}{\theta^8} + \frac{10\sin \theta}{\theta^9} \right), \\
\gamma &= \frac{\csc \theta}{\theta} \left(\frac{85399 - 88234\cos \theta}{181440} + \frac{107 + 22\cos \theta}{360\theta^2} + \frac{79 + 86\cos \theta}{60\theta^4} + \frac{29 + 34\cos \theta}{3\theta^6} + \right. \\
&\quad \left. \frac{98 + 112\cos \theta}{\theta^8} - \frac{210\sin \theta}{\theta^9} \right), \\
\delta &= \frac{\csc \theta}{\theta} \left(\frac{2773 - 91\cos \theta}{11340} - \frac{17 - 11\cos \theta}{315\theta^2} - \frac{8 - 8\cos \theta}{15\theta^4} - \frac{16 + 8\cos \theta}{3\theta^6} - \frac{64 + 56\cos \theta}{\theta^8} + \right. \\
&\quad \left. \frac{120\sin(\theta)}{\theta^9} \right),
\end{aligned}$$

$$\eta = \frac{\csc \theta}{\theta} \left(\frac{85399 - 88234 \cos \theta}{181440} + \frac{107 + 22 \cos \theta}{360 \theta^2} + \frac{79 + 86 \cos \theta}{60 \theta^4} + \frac{29 + 34 \cos \theta}{3 \theta^6} + \frac{98 + 112 \cos \theta}{\theta^8} - \frac{210 \sin \theta}{\theta^9} \right),$$

$$\lambda = \frac{\csc \theta}{\theta} \left(\frac{44117 - 78095 \cos \theta}{90720} - \frac{11 + 175 \cos \theta}{180 \theta^2} - \frac{43 + 50 \cos \theta}{30 \theta^4} - \frac{34 + 50 \cos \theta}{3 \theta^6} - \frac{112 + 140 \cos \theta}{\theta^8} + \frac{252 \sin \theta}{\theta^9} \right).$$

If $\theta \rightarrow 0$ then

$$(\alpha, \beta, \gamma, \delta, \eta, \lambda) \rightarrow \left(\frac{1}{39916800}, \frac{2036}{39916800}, \frac{152637}{39916800}, \frac{2203488}{39916800}, \frac{9738114}{39916800}, \frac{15724248}{39916800} \right).$$

And the consistency relation of non-polynomial is reduced to consistency relation of the eleventh polynomial spline functions derived in [5]. The local truncation error corresponding to the method equation (5) can be obtained as:

$$\begin{aligned} t_i = & (1 - (2\alpha + 2\beta + 2\gamma + 2\delta + 2\eta + \lambda)h^{10}y_i^{(10)} + (\frac{5}{12} - (25\alpha + 16\beta + 9\gamma + 4\delta + \eta)h^{12}y_i^{(12)} \\ & + (\frac{1}{12} - \frac{1}{12}(625\alpha + 256\beta + 81\gamma + 16\delta + \eta))h^{14}y_i^{(14)} \\ & + (\frac{43}{4032} - \frac{1}{360}(15625\alpha + 4096\beta + 729\gamma + 64\delta + \eta))h^{16}y_i^{(16)} \\ & + (\frac{713}{725760} - \frac{1}{20160}(390625\alpha + 65536\beta + 6561\gamma + 256\delta + \eta))h^{18}y_i^{(18)} \\ & + (\frac{317}{4561920} - \frac{1}{1814400}(9765625\alpha + 1048576\beta + 59049\gamma + 1024\delta + \eta))h^{20}y_i^{(20)} \\ & + (\frac{1}{1816214400}(7141 - \frac{91}{12}(244140625\alpha + 16777216\beta + 531441\gamma + 4096\delta + \eta)))h^{22}y_i^{(22)} \end{aligned} \quad i = 6, 7, \dots, n-6 \quad (6)$$

Case(1): Second-order method

If me choose $\alpha = \frac{1}{39916800}$, $\beta = \frac{2036}{39916800}$, $\gamma = \frac{152637}{39916800}$, $\delta = \frac{2203488}{39916800}$

$\eta = \frac{9738114}{39916800}$ and $\lambda = \frac{15724248}{39916800}$ the truncation errors(6) with be $O(h^{12})$.

Case(2): Fourth-order method

If me choose $\alpha = 0, \beta = 0, \gamma = 0, \delta = 0, \eta = \frac{5}{12}$ and $\lambda = \frac{1}{6}$ the truncation errors(6) with be $O(h^{14})$.

Case (3): Sixth-order method

If me choose $\alpha = 0, \beta = 0, \gamma = 0, \delta = \frac{7}{144}, \eta = \frac{2}{9}$ and $\lambda = \frac{11}{24}$ the truncation errors(6) with be $O(h^{16})$.

Case (4): Eight-order method

If me choose $\alpha = 0, \beta = 0, \gamma = \frac{17}{12096}, \delta = \frac{9}{224}, \eta = \frac{109}{448}$ and $\lambda = \frac{1301}{3024}$ the truncation errors(6) with be $O(h^{18})$.

Case (5): Tenth-order method

If me choose $\alpha = 0, \beta = \frac{1}{362880}, \gamma = \frac{251}{181440}, \delta = \frac{913}{22680}, \eta = \frac{44117}{181440}$ and $\lambda = \frac{15619}{36288}$ the truncation errors(6) with be $O(h^{20})$.

Case (6): Twelfth-order method

If me choose $\alpha = \frac{1}{47900160}, \beta = \frac{61}{23950080}, \gamma = \frac{22103}{15966720}, \delta = \frac{11477}{285120}$ and $\eta = \frac{215687}{887040}, \lambda = \frac{1718069}{3991680}$ the truncation errors(6) with be $O(h^{22})$.

3 Development of the boundary formulas

Liner system equation (5) consist of $(n - 1)$ unknown, so that to obtain unique solution we need tenth more equations to be associate with equation (5) so that we can develop the boundary formulas of different orders, but for sake of briefness here we develop the tenth order boundary formulas so that we define the following identity:

$$w_0' y_0 + \sum_{i=0}^6 a_i' y_{i+\frac{1}{2}} + c' h y_0' + d' h^2 y_0'' + e' h^3 y_0''' + u' h^4 y_0^{(4)} = h^{10} \sum_{i=0}^8 b_i' y_{i+\frac{1}{2}}^{(10)} \quad (7)$$

$$w_0'' y_0 + \sum_{i=0}^7 a_i'' y_{i+\frac{1}{2}} + c'' h y_0' + d'' h^2 y_0'' + e'' h^3 y_0''' + u'' h^4 y_0^{(4)} = h^{10} \sum_{i=0}^9 b_i'' y_{i+\frac{1}{2}}^{(10)} \quad (8)$$

$$w_0''' y_0 + \sum_{i=0}^8 a_i''' y_{i+\frac{1}{2}} + c''' h y_0' + d''' h^2 y_0'' + e''' h^3 y_0''' + u''' h^4 y_0^{(4)} = h^{10} \sum_{i=0}^{10} b_i''' y_{i+\frac{1}{2}}^{(10)} \quad (9)$$

$$w_0^\circ y_0 + \sum_{i=0}^9 a_i^\circ y_{i+\frac{1}{2}} + c^\circ h y_0' + d^\circ h^2 y_0'' + e^\circ h^3 y_0''' + u^\circ h^4 y_0^{(4)} = h^{10} \sum_{i=0}^{11} b_i^\circ y_{i+\frac{1}{2}}^{(10)} \quad (10)$$

$$w_0^{\circ\circ} y_0 + \sum_{i=0}^{10} a_i^{\circ\circ} y_{i+\frac{1}{2}} + c^{\circ\circ} h y_0' + d^{\circ\circ} h^2 y_0'' + e^{\circ\circ} h^3 y_0''' + u^{\circ\circ} h^4 y_0^{(4)} = h^{10} \sum_{i=0}^{12} b_i^{\circ\circ} y_{i+\frac{1}{2}}^{(10)} \quad (11)$$

$$w_0^\bullet y_n + \sum_{i=0}^{10} a_i^\bullet y_{i+n-\frac{21}{2}} + c^\bullet h y_n' + d^\bullet h^2 y_n'' + e^\bullet h^3 y_n''' + u^\bullet h^4 y_n^{(4)} = h^{10} \sum_{i=0}^{12} b_i^\bullet y_{i+n-\frac{25}{2}}^{(10)} \quad (12)$$

$$w_0^* y_n + \sum_{i=0}^9 a_i^* y_{i+n-\frac{19}{2}} + c^* h^2 y_n' + d^* h^2 y_n'' + e^* h^3 y_n''' + u^* h^4 y_n^{(4)} = h^{10} \sum_{j=0}^{11} b_i^* y_{i+n-\frac{23}{2}}^{(10)} \quad (13)$$

$$\bar{w}_0 y_n + \sum_{i=0}^8 \bar{a}_i y_{i+n-\frac{17}{2}} + \bar{c} h y_n' + \bar{d} h^2 y_n'' + \bar{e} h^3 y_n''' + \bar{u} h^4 y_n^{(4)} = h^{10} \sum_{i=0}^{10} \bar{b}_i y_{i+n-\frac{21}{2}}^{(10)} \quad (14)$$

$$\breve{w}_0 y_n + \sum_{i=0}^7 \breve{a}_i y_{i+n-\frac{15}{2}} + \breve{c} h y_n' + \breve{d} h^2 y_n'' + \breve{e} h^3 y_n''' + \breve{u} h^4 y_n^{(4)} = h^{10} \sum_{i=0}^9 \breve{b}_i y_{i+n-\frac{19}{2}}^{(10)} \quad (15)$$

$$\widetilde{w}_0 y_n + \sum_{i=0}^6 \widetilde{a}_i y_{i+n-\frac{13}{2}} + \widetilde{c} h y_n' + \widetilde{d} h^2 y_n'' + \widetilde{e} h^3 y_n''' + \widetilde{u} h^4 y_n^{(4)} = h^{10} \sum_{i=0}^8 \widetilde{b}_i y_{i+n-\frac{17}{2}}^{(10)} \quad (16)$$

Where all of the coefficients are arbitrary parameters to be determined.

4 Nonpolynomial spline solution

The methods (5) along with boundary condition (7)-(16) when we ignore the truncation errors in (6) give a system of linear equations.

Considering $Y = [y_{\frac{1}{2}}, y_{\frac{3}{2}}, \dots, y_{\frac{n-1}{2}}]^T$ and $C = [c_{\frac{1}{2}}, c_{\frac{3}{2}}, \dots, c_{\frac{n-1}{2}}]^T$, This system can be written the following matrix equation:

$$(A + h^{10} BF)Y = C$$

Where

$$A = \begin{bmatrix} a_0^{\cdot} & a_1^{\cdot} & a_2^{\cdot} & a_3^{\cdot} & a_4^{\cdot} & a_5^{\cdot} & a_6^{\cdot} \\ a_0^{\prime} & a_1^{\prime} & a_2^{\prime} & a_3^{\prime} & a_4^{\prime} & a_5^{\prime} & a_6^{\prime} & a_7^{\prime} \\ a_0^{\prime\prime} & a_1^{\prime\prime} & a_2^{\prime\prime} & a_3^{\prime\prime} & a_4^{\prime\prime} & a_5^{\prime\prime} & a_6^{\prime\prime} & a_7^{\prime\prime} & a_8^{\prime\prime} \\ a_0^{\prime\prime\prime} & a_1^{\prime\prime\prime} & a_2^{\prime\prime\prime} & a_3^{\prime\prime\prime} & a_4^{\prime\prime\prime} & a_5^{\prime\prime\prime} & a_6^{\prime\prime\prime} & a_7^{\prime\prime\prime} & a_8^{\prime\prime\prime} \\ a_0^{\circ} & a_1^{\circ} & a_2^{\circ} & a_3^{\circ} & a_4^{\circ} & a_5^{\circ} & a_6^{\circ} & a_7^{\circ} & a_8^{\circ} & a_9^{\circ} \\ a_0^{\circ\circ} & a_1^{\circ\circ} & a_2^{\circ\circ} & a_3^{\circ\circ} & a_4^{\circ\circ} & a_5^{\circ\circ} & a_6^{\circ\circ} & a_7^{\circ\circ} & a_8^{\circ\circ} & a_9^{\circ\circ} & a_{10}^{\circ\circ} \\ 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 \\ \vdots & \vdots \\ 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 \\ a_{10}^{\cdot} & a_9^{\cdot} & a_8^{\cdot} & a_7^{\cdot} & a_6^{\cdot} & a_5^{\cdot} & a_4^{\cdot} & a_3^{\cdot} & a_2^{\cdot} & a_1^{\cdot} & a_0^{\cdot} \\ a_9^* & a_8^* & a_7^* & a_6^* & a_5^* & a_4^* & a_3^* & a_2^* & a_1^* & a_0^* & \\ \bar{a}_8 & \bar{a}_7 & \bar{a}_6 & \bar{a}_5 & \bar{a}_4 & \bar{a}_3 & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 & \\ \bar{a}_7 & \bar{a}_6 & \bar{a}_5 & \bar{a}_4 & \bar{a}_3 & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 & \\ \bar{a}_6 & \bar{a}_5 & \bar{a}_4 & \bar{a}_3 & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 & \\ \end{bmatrix}$$

$$B = \begin{bmatrix} b_0^{\cdot} & b_1^{\cdot} & b_2^{\cdot} & b_3^{\cdot} & b_4^{\cdot} & b_5^{\cdot} & b_6^{\cdot} & b_7^{\cdot} & b_8^{\cdot} \\ b_0^{\prime} & b_1^{\prime} & b_2^{\prime} & b_3^{\prime} & b_4^{\prime} & b_5^{\prime} & b_6^{\prime} & b_7^{\prime} & b_8^{\prime} & b_9^{\prime} \\ b_0^{\prime\prime} & b_1^{\prime\prime} & b_2^{\prime\prime} & b_3^{\prime\prime} & b_4^{\prime\prime} & b_5^{\prime\prime} & b_6^{\prime\prime} & b_7^{\prime\prime} & b_8^{\prime\prime} & b_9^{\prime\prime} & b_{10}^{\prime\prime} \\ b_0^{\circ} & b_1^{\circ} & b_2^{\circ} & b_3^{\circ} & b_4^{\circ} & b_5^{\circ} & b_6^{\circ} & b_7^{\circ} & b_8^{\circ} & b_9^{\circ} & b_{10}^{\circ} & b_{11}^{\circ} \\ b_0^{\circ\circ} & b_1^{\circ\circ} & b_2^{\circ\circ} & b_3^{\circ\circ} & b_4^{\circ\circ} & b_5^{\circ\circ} & b_6^{\circ\circ} & b_7^{\circ\circ} & b_8^{\circ\circ} & b_9^{\circ\circ} & b_{10}^{\circ\circ} & b_{11}^{\circ\circ} & b_{12}^{\circ\circ} \\ \alpha & \beta & \gamma & \delta & \eta & \lambda & \eta & \delta & \gamma & \beta & \alpha & \\ \vdots & \vdots \\ \alpha & \beta & \gamma & \delta & \eta & \lambda & \eta & \delta & \gamma & \beta & \alpha & \\ b_{12}^{\bullet} & b_{11}^{\bullet} & b_{10}^{\bullet} & b_9^{\bullet} & b_8^{\bullet} & b_7^{\bullet} & b_6^{\bullet} & b_5^{\bullet} & b_4^{\bullet} & b_3^{\bullet} & b_2^{\bullet} & b_1^{\bullet} & b_0^{\bullet} \\ b_{11}^* & b_{10}^* & b_9^* & b_8^* & b_7^* & b_6^* & b_5^* & b_4^* & b_3^* & b_2^* & b_1^* & b_0^* & \\ \bar{b}_{10} & \bar{b}_9 & \bar{b}_8 & \bar{b}_7 & \bar{b}_6 & \bar{b}_5 & \bar{b}_4 & \bar{b}_3 & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 & \\ \bar{b}_9 & \bar{b}_8 & \bar{b}_7 & \bar{b}_6 & \bar{b}_5 & \bar{b}_4 & \bar{b}_3 & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 & \\ \bar{b}_8 & \bar{b}_7 & \bar{b}_6 & \bar{b}_5 & \bar{b}_4 & \bar{b}_3 & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 & \\ \end{bmatrix}$$

$$F = \text{diag}(f_i), i = 1, 2, 3, \dots, n-1.$$

The vector C is defined by

$$\begin{aligned}
 c_{\frac{1}{2}} &= -w_0^y y_0 - c^y h y_0^y - d^y h^2 y_0^y - e^y h^3 y_0^y - u^y h^4 y_0^{(4)} + h^{10} \sum_{i=0}^8 b_i g_{i+\frac{1}{2}}, \\
 c_{\frac{3}{2}} &= -w_0^y y_0 - c^y h y_0^y - d^y h^2 y_0^y - e^y h^3 y_0^y - u^y h^4 y_0^{(4)} + h^{10} \sum_{i=0}^9 b_i g_{i+\frac{1}{2}}, \\
 c_{\frac{5}{2}} &= -w_0^y y_0 - c^y h y_0^y - d^y h^2 y_0^y - e^y h^3 y_0^y - u^y h^4 y_0^{(4)} + h^{10} \sum_{i=0}^{10} b_i g_{i+\frac{1}{2}}, \\
 c_{\frac{7}{2}} &= -w_0^y y_0 - c^y h y_0^y - d^y h^2 y_0^y - e^y h^3 y_0^y - u^y h^4 y_0^{(4)} + h^{10} \sum_{i=0}^{11} b_i g_{i+\frac{1}{2}}, \\
 c_{\frac{9}{2}} &= -w_0^y y_0 - c^y h y_0^y - d^y h^2 y_0^y - e^y h^3 y_0^y - u^y h^4 y_0^{(4)} + h^{10} \sum_{i=0}^{12} b_i g_{i+\frac{1}{2}}, \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 c_{\frac{i-1}{2}} &= h^{10} (\alpha g_{i-\frac{11}{2}} + \beta g_{i-\frac{9}{2}} + \gamma g_{i-\frac{7}{2}} + \delta g_{i-\frac{5}{2}} + \eta g_{i-\frac{3}{2}} + \lambda g_{i-\frac{1}{2}} + \eta g_{i+\frac{1}{2}} + \delta g_{i+\frac{3}{2}} + \\
 &\quad \gamma g_{i+\frac{5}{2}} + \beta g_{i+\frac{7}{2}} + \alpha g_{i+\frac{9}{2}}), \quad i = 6, 7, \dots, (n-6) \\
 &\vdots \\
 &\vdots \\
 c_{\frac{n-9}{2}} &= -w_0^y y_n - c^y h^2 y_n^y - d^y h^2 y_n^y - e^y h^3 y_n^y - u^y h^4 y_n^{(4)} + h^{10} \sum_{i=0}^{12} b_i^y g_{i+n-\frac{25}{2}}, \\
 c_{\frac{n-7}{2}} &= -w_0^y y_n - c^y h^2 y_n^y - d^y h^2 y_n^y - e^y h^3 y_n^y - u^y h^4 y_n^{(4)} + h^{10} \sum_{i=0}^{11} b_i^y g_{i+n-\frac{23}{2}}, \\
 c_{\frac{n-5}{2}} &= -\bar{w}_0 y_n - \bar{c} h y_n^y - \bar{d} h^2 y_n^y - \bar{e} h^3 y_n^y - \bar{u} h^4 y_n^{(4)} + h^{10} \sum_{i=0}^{10} \bar{b}_i g_{i+n-\frac{21}{2}}, \\
 c_{\frac{n-3}{2}} &= -\bar{w}_0 y_n - \bar{c} h y_n^y - \bar{d} h^2 y_n^y - \bar{e} h^3 y_n^y - \bar{u} h^4 y_n^{(4)} + h^{10} \sum_{i=0}^9 \bar{b}_i g_{i+n-\frac{19}{2}}, \\
 c_{\frac{n-1}{2}} &= -\bar{w}_0 y_n - \bar{c} h y_n^y - \bar{d} h^2 y_n^y - \bar{e} h^3 y_n^y - \bar{u} h^4 y_n^{(4)} + h^{10} \sum_{i=0}^8 \bar{b}_i g_{i+n-\frac{17}{2}},
 \end{aligned}$$

5 NUMERICAL RESULTS

In this section the presented method are applied to the following test problems if choosing

$$(\alpha, \beta, \gamma, \delta, \eta, \lambda) = \left(\frac{1}{47900160}, \frac{61}{23950080}, \frac{22103}{15966720}, \frac{11477}{285120}, \frac{215687}{887040}, \frac{1718069}{3991680} \right)$$

we obtained the method of order $O(h^{22})$ respectively.

Example 1. We Consider the following boundary-value problem

$$y^{(10)}(x) - (x^2 - 2x)y(x) = 10 \cos x - (x-1)^3 \sin x, \quad -1 \leq x \leq 1,$$

$$\begin{aligned}
y(-1) &= 2 \sin(1), y(1) = 0, \\
y'(-1) &= -2 \cos(1) - \sin(1), y'(1) = \sin(1), \\
y''(-1) &= 2 \cos(1) - 2 \sin(1), y''(1) = 2 \cos(1), \\
y'''(-1) &= 2 \cos(1) + 3 \sin(1), y'''(1) = -3 \sin(1), \\
y^{(4)}(-1) &= -4 \cos(1) + 2 \sin(1), y^{(4)}(1) = -4 \cos(1).
\end{aligned} \tag{17}$$

The exact solution for this problem is $y(x) = (x-1)\sin x$. We solved this example by different values of $h = \frac{1}{14}, \frac{1}{28}, \frac{1}{42}, \frac{1}{56}$. The maximum absolute errors associated with y_i for the system (17) are summarized in Table 1 and compared with [5].

Table 1 : Maximum absolute errors of Example 1

n	Our method	Method in[5]
14	3.84×10^{-11}	5.96×10^{-6}
28	8.48×10^{-11}	7.99×10^{-7}
42	1.64×10^{-12}	1.72×10^{-7}
56	3.51×10^{-13}	3.73×10^{-8}

Example 2. We Consider the following boundary-value problem

$$\begin{aligned}
y^{(10)}(x) - xy(x) &= -(89 + 21x + x^2 - x^3)e^x, \quad -1 \leq x \leq 1, \\
y(-1) &= 0, y(1) = 0, \\
y'(-1) &= \frac{2}{e}, y'(1) = -2e, \\
y''(-1) &= \frac{2}{e}, y''(1) = -6e, \\
y'''(-1) &= 0, y'''(1) = -12e, \\
y^{(4)}(-1) &= \frac{-4}{e}, y^{(4)}(1) = -20e.
\end{aligned} \tag{18}$$

The exact solution for this problem is $y(x) = (1-x^2)e^x$. We solved this example by different values of $h = \frac{1}{9}$. The maximum absolute errors associated with y_i for the system (18) are summarized in Table 2 and compared with [5].

Table 2 : Maximum absolute errors of Example 2

n	Our method	Method in[5]
9	1.75×10^{-12}	3.28×10^{-6}

Example 3. We Consider the following boundary-value problem

$$\begin{aligned}
y^{(10)}(x) + y(x) &= -10(2x \sin x - 9 \cos x), \quad -1 \leq x \leq 1, \\
y(-1) &= 0, y(1) = 0, \\
y'(-1) &= -2 \cos(1), y'(1) = 2 \cos(1), \\
y''(-1) &= 2 \cos(1) - 4 \sin(1), y''(1) = 2 \cos(1) - 4 \sin(1), \\
y'''(-1) &= 6 \cos(1) + 6 \sin(1), y'''(1) = -6 \cos(1) - 6 \sin(1),
\end{aligned}$$

$$y^{(4)}(-1) = -12 \cos(1) + 8 \sin(1), y^{(4)}(1) = -12 \cos(1) + 8 \sin(1). \quad (19)$$

The exact solution for this problem is $y(x) = (x^2 - 1) \cos x$. We solved this example by different values of $h = \frac{1}{16}$. The maximum absolute errors associated with y_i for the system (19) are summarized in Table 3 and compared with [5].

Table 3 : Maximum absolute errors of Example 3

n	Our method	Method in[5]
16	3.25×10^{-14}	8.85×10^{-8}

Conclusion

We approximate solution of the tenth-order linear boundary-value problems by using non-polynomial spline. The new methods enable us to approximate the solution at every point of the range of integration. Tables 1-3 show that our methods produced better in the sense that $\max |e_i|$ is minimum in comparison with the methods developed in [5].

REFERENCES

- [1] S. Chandrasekhar. Hydrodynamic and Hydromagnetic Stability. Clarendon press, New York, 1981. Oxford, 1961.
- [2] E. Twizell, A. Boutayeb, K. Djidjeli. Numerical methods for eighth, tenth and twelfth-order eigenvalue problems arising in thermal instability. Advances in Computational Mathematics, 1994, 2: 407–436.
- [3] S. Siddiqi, E. Twizell. Spline solutions of liner tenth-order boundary value problems. International Journal of Computer International Journal of Computer, 1998, 68: 345–362.
- [4] S. Siddiqi, G. Akram. Solution of 10th-order boundary value problems using non-polynomial spline technique. Applied Mathematics and Computation, 2007, 190: 641–651.
- [5] S. Siddiqi, G. Akram. Solution of tenth-order boundary value problems using eleventh degree spline. Applied Mathematics and Computation, 2007, 185: 115–127.
- [6] J. Rashidinia, R. Jalilian, K. Farajeyan. Spline approximate solution of eighth-order boundary-value problems. International Journal of Computer Mathematics, 2009, 86: 1319–1333.
- [7] Siraj-Ul-Islam, Khan M.A. A numerical method based on polynomial sextic spline functions for the solution of special fifth-order boundary-value problems, Appl. Math. Comput. 2006, 181:336-361.
- [8] Scott M.R., Watts H.A., "A Systematized Collection of Codes for Solving Two-point bvp's, Numerical Methods for Differential Systems," Academic Press, 1976.