

Modification of Centralized Resource Allocation Model with Variable Return to Scale Using Facet Analysis

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ABSTRACT

A centralized decision maker (DM) as supervises of its decision making units (DMUs), interest to minimizing the total input consumption and maximizing the total output production. This can be done by maximizing the efficiency of individual units using by conventional DEA models. Lozano and Villa (2004) use these models for increasing the efficiency of each DMU and minimizing the total input or maximizing the total output, at the same time. This paper modified the models which used the above process by restricted their free variables based on Daneshvar (2009). The advantages of this modification are considered by examples.

KEYWORDS: Data envelopment analysis, Centralized resource allocation, Facet analysis

1. INTRODUCTION

Clearly, centralized resource allocation models in input orientation (CRA-I) which presented by Lozano and Villa in 2004 are a type of data envelopment analysis models with variable return to scale. This paper consider two models, the first model seeks radial reductions of the total consumption of every input while the second model seeks separate reductions levels for the total amount of each input according to a preference structure. With this manner, DM can reduce sum of the inputs which consumed by DMUs without reducing in the summation of outputs produced by them.

As is known, the BCC Model has variable return to scale but sometimes zero weights make some problems for evaluation of DMUs. To prevent of zero weights Daneshvar modified the BCC model using facet analysis in 2009. This is done by identification of hyperplanes which make weak parts of production possibility set and avoidance of them by restricted the free variable of BCC model.

This paper, try to modified the centralized resource allocation models based on Daneshvar 2009. So, radial centralized resource allocation describe in the next section. In section 3 the modification of BCC model is present. Modification of centralized resource allocation models illustrate in section 4. In section 5 the numerical results are present. At last, section 6 includes conclusions and some remarks.

2. Radial centralized resource allocation/Input-oriented

As it is usual with radial models, there are two phases. In the first phase, an equi- proportional reduction along all input dimensions is sought while, in the second phase, additional reduction of any input and/or expansion of any output are pursued. Here we have two basic differences with conventional DEA models:

1. Instead of solving an independent LP model for projecting each DMU on the efficiency frontier, all DMUs are simultaneously projected.

2. Instead of reducing the inputs for each DMU, the aim is to reduce the total input consumption of the DMUs.

Let j, r = 1, ..., n be indexes for DMUs; i = 1, ..., m, be index for inputs; k = 1, ..., p, be index for outputs, x_{ij} , amount of input *i* consumed by DMU_j; y_{kj} , quantity of output *k* produced by DMU_j; θ , radial contraction of total input vector; s_i , slack along the input dimension *i*; t_k , additional increase along the output dimension *k* ($\lambda_{1r}, \lambda_{2r}, ..., \lambda_{nr}$) vector for projecting DMU_r. The phase I model is:

Model phase I / Radial / Input- oriented

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 $\min \theta$

$$\begin{split} \sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} x_{ij} &\leq \theta \sum_{j=1}^{n} x_{ij} & i = 1, 2, ..., m \\ \sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} y_{kj} &\geq \sum_{j=1}^{n} y_{kj} & k = 1, 2, ..., p \\ \sum_{j=1}^{n} \lambda_{jr} &= 1 & r = 1, 2, ..., n \\ \lambda_{jr} &\geq 0 & j, r = 1, 2, ..., n \\ \theta & free & (1) \end{split}$$

This is an LP model with $n^2 + 1$ variables and m + p + n constraints. Let θ^* be the optimum of the previous model, then the phase II model can be formulated as:

Model phase II/ Radial/ Input- oriented

$$\max \sum_{i=1}^{m} s_{i} + \sum_{k=1}^{p} t_{k}$$

$$\sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} x_{ij} = \theta^{*} \sum_{j=1}^{n} x_{ij} - s_{i} \qquad i = 1,...,m$$

$$\sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} y_{kj} = \sum_{j=1}^{n} y_{kj} + t_{k} \qquad k = 1,...,p$$

$$\sum_{j=1}^{n} \lambda_{jr} = 1 \qquad r = 1,...,n$$

$$\lambda_{jr} \ge 0 \qquad j, r = 1,...,n$$

$$s_{i} \ge 0 \qquad i = 1,...,m$$

$$t_{k} \ge 0 \qquad k = 1,...,p$$
(2)

Once the / Input- oriented model is solved, the corresponding vector $(\lambda_{1r}^*, \lambda_{2r}^*, ..., \lambda_{nr}^*)$ defines for each DMU_r the operating point at which it should aim. The inputs and outputs of each such point can be computed as

$$\hat{x}_{ir} = \sum_{j=1}^{n} \lambda^{*}_{jr} x_{ij} \qquad i = 1,...,m$$
$$\hat{y}_{kr} = \sum_{j=1}^{n} \lambda^{*}_{jr} y_{kj} \qquad k = 1,...,p \quad (3)$$

It is interesting to note the following proposition.

Proposition 1

For any DMU_r, the correspond operating point which it is projected by model phase II /Radial/Input- oriented $(\hat{x}_{1r}, \hat{x}_{2r}, ..., \hat{x}_{mr}, \hat{y}_{1r}, \hat{y}_{2r}, ..., \hat{y}_{pr})$ is pareto efficient.

The dual form of model (1) as follows:

Model dual /phase I / Radial/ Input- oriented

$$\max \sum_{k=1}^{m} u_{k} \sum_{j=1}^{n} y_{kj} + \sum_{r=1}^{n} \xi_{r}$$

$$\sum_{i=1}^{m} v_{i} \sum_{j=1}^{n} x_{ij} = 1$$

$$\sum_{k=1}^{p} u_{k} y_{kj} - \sum_{i=1}^{m} v_{i} x_{ij} + \xi_{r} \le 0 \quad j, r = 1, ..., n$$

$$v_{i} \ge 0 \qquad i = 1, ..., m$$

$$u_{k} \ge 0 \qquad k = 1, ..., p$$

$$\xi_{r} \quad \text{free} \qquad r = 1, ..., n \quad (4)$$

In section 4, dual form will used for modification.

3. Modification of BCC model using bounded free variable

In Daneshvar (2009) current BCC models have been modified by bounded free variable and adding the constraint $u_o \leq \varepsilon$ instead of m + p constraint $U \geq 1\varepsilon, V \geq 1\varepsilon$, for avoidance of zero weights. In this approach first we solve (5) for all DMUs.

$$\overline{u_0} = \min u_0$$

$$\sum_{r=1}^{s} u_r y_{ro} + u_0 = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + \sum_{i=1}^{m} v_i x_{ij} + u_0 \le 0 \quad j = 1, ..., n$$

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$u_r \ge 0 \quad r = 1, ..., s$$

$$v_i \ge 0 \quad i = 1, ..., m$$

$$u_o \quad \text{free} \quad (5)$$

If the optimal solution of this model denoted by $\bar{u_0}$, the value of ε can be defined as follows:

 $\varepsilon = \max{\{\overline{u}_0 ; \overline{u}_0 \neq 1; \text{ tecnical efficient } DMU_o\}}$ (6)

Then ε has been considered as upper bound of free variable u_{o} , and the BCC model modified as:

$$Max \sum_{r=1}^{s} u_{r} y_{ro} + u_{0}$$

$$\sum_{r=1}^{s} u_{r} y_{rj} + \sum_{i=1}^{m} v_{i} x_{ij} + u_{0} \le 0 \quad j = 1, ..., n$$

$$\sum_{i=1}^{m} v_{i} x_{io} = 1$$

$$u_{r} \ge 0 \quad r = 1, ..., s$$

$$v_{i} \ge 0 \quad i = 1, ..., m$$

$$u_{0} \le \varepsilon \quad (5)$$

4. Modification of centralized resource allocation (VRS)/Radial/Input-oriented

In this section the radial centralized resource allocation models with variable returns to scale modified based on previous section. For do this, first the value of ε obtain by using (6), (7) for all efficient DMUs and then consider as upper bound of free variable in centralized resource allocation models. Here the aim of modified model is reduction the sum inputs without any decreasing in the summation of outputs in the absence of weak efficiency frontier. For this purpose, we have

Modified centralized resource allocation model (VRS) /Radial/Input-oriented

$$\max \sum_{k=1}^{p} u_{k} \sum_{j=1}^{n} y_{kj} + \sum_{r=1}^{n} \xi_{r}$$

$$\sum_{i=1}^{m} v_{i} \sum_{j=1}^{n} x_{ij} = 1$$

$$\sum_{k=1}^{p} u_{k} y_{kj} - \sum_{i=1}^{m} v_{i} x_{ij} + \xi_{r} \le 0 \quad j, r = 1, ..., n$$

$$\xi_{r} \le \varepsilon \qquad r = 1, ..., n$$

$$v_{i} \ge 0 \qquad i = 1, ..., m$$

$$u_{k} \ge 0 \qquad k = 1, ..., p \quad (8)$$

Now, the dual of above model is modified phase I:

Modified phase I/Radial/Input-oriented

$$\min \left(\theta + \varepsilon \sum_{r=1}^{n} \partial_{r} \right)$$

$$\sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} x_{ij} \leq \theta \sum_{j=1}^{n} x_{ij} \quad i = 1, ..., m$$

$$\sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} y_{kj} \geq \sum_{j=1}^{n} y_{kj} \quad k = 1, ..., p$$

$$\sum_{j=1}^{n} \lambda_{jr} + \partial_{r} = 1 \qquad r = 1, ..., n$$

$$\theta \qquad \text{free}$$

$$\lambda_{jr} \geq 0 \qquad j, r = 1, ..., n$$

$$\partial_{r} \geq 0 \qquad r = 1, ..., n$$

$$(9)$$

This is an LP model with $n^2 + 1 + n$ variables and m + p + n constraints. Let θ^* be the optimum of the previous model, then the modified phase II model can be formulated as:

Modified phase II /Radial/Input-oriented

$$\max \sum_{i=1}^{m} s_{i} + \sum_{k=1}^{p} t_{k}$$

$$\sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} x_{ij} + s_{i} = \theta^{*} \sum_{j=1}^{n} x_{ij} \quad i = 1, ..., m$$

$$\sum_{r=1}^{n} \sum_{j=1}^{n} \lambda_{jr} y_{kj} - t_{k} = \sum_{j=1}^{n} y_{kj} \quad k = 1, ..., p$$

$$\sum_{j=1}^{n} \lambda_{jr} + \partial_{r} = 1 \quad r = 1, ..., n \quad , \lambda_{jr} \ge 0 \quad j, r = 1, ..., n$$

$$s_{i} \ge 0 \quad i = 1, ..., m \quad , t_{k} \ge 0 \quad k = 1, ..., p \quad (10)$$

Once modified phase II model is solved, the corresponding vector $(\lambda_{1r}^*, \lambda_{2r}^*, ..., \lambda_{nr}^*)$ defines for each DMU_r the operating point at which it should aim. The inputs and outputs of each such point can be computed as:

$$x^{\varepsilon}_{ir} = \sum_{j=1}^{n} \lambda^{*}_{jr} x_{ij}$$
$$y^{\varepsilon}_{kr} = \sum_{j=1}^{n} \lambda^{*}_{jr} y_{kj} \qquad (11)$$

5. Numerical examples

Example1.

Table (1) shows the change in total of inputs and total of outputs after using the various methods for 7 DMUs with one input and one output. In this table DM first used BBC-I for maximizing the individual efficiency score of DMUs, then the projection of DMUs on efficient frontier are consider. After that CRA-I models is used on mentioned data and then the modified models used for same DMUs, with respect of \overline{u}_o values. The total of inputs and the total of outputs are compute for all methods in least row of Table (1).

Table (1): data and results for example 1(one-input, one-output)											
	Exi	sting			CRA	A-I	Modified by				
									$u_o \leq \varepsilon = 0.8$		
DMU	x	У	$oldsymbol{ heta}^*$ BCC	\overline{u}_{o}	x^*	y^{*}	x	ŷ	x^{ε}	y^{ε}	
А	3	33	1	0.8	3	3	4	8	12.5995	25.199	
В	4	8	1	0	4	8	4	8	0	0	
С	5	5	0.68	N.F	3.4	5	3.6	6	0	0	
D	5	10	1	-3.00	5	10	4	8	0	0	
Е	6	8	0.6667	N.F	4	8	4	8	5	10	
F	7	11	1	$-\infty$	7	11	4	8	5	10	
G	8	9	0.5625	N.F	4.5	9	4	8	5	10	
sum	38	54	-		30.9	54	27.6	54	27.5995	55.199	

For this data, ε have been computed using (6), (7).

$$\varepsilon = \max\{0.8, 0, -3, -\infty\} = 0.8$$

Hear, it can be seen that, after modification the sum of outputs have been increased.

Example2.

This example contains 4 DMUs with two-inputs and one-output. As seen in Table (2), these DUMs consider in a manner that all of them are efficient, then the BCC-I and CRA-I models cannot create any decreasing in total of inputs, but modified model has been successful to attaining this aim. The upper bound for free variable is computed as: $\varepsilon = \max\{-\infty, -1.6667, -1, 0\} = 0$

	Table (2): example 2														
		Existing					BCC-I CRA-I					Modified by			
														$u_o \leq \varepsilon = 0$	
	DMU	x_1	<i>x</i> ₂	y_1	${oldsymbol{ heta}}^*{}_{\scriptscriptstyle BCC}$	\overline{u}_{o}	x^{*_1}	x^* 2	$y^{*_{1}}$	\hat{x}_1	\hat{x}_2	\hat{y}_1	x^{ε}_{1}	x^{ε} 2	y^{ε} 1
Г	А	7	2	3	1	$-\infty$	7	2	3	3	5	2	3	5	2
	В	3	2	2	1	-1.6667	3	5	2	7	1	1.5	6.6152	2.28875	2.9038
	С	5	3.5	2.5	1	-1	5	3.5	2.5	5	3.5	2.5	7	2	3
	D	7	1	1.5	1	0	7	1	1.5	7	2	3	2.5578	7.308	1.0962
Г	Sum	22	11.5	9	-	-	22	11.5	9	22	11.5	9	19.173	10.01955	9

Example 3.

Table (3) shows the same procedure for 10 DMUs with two-inputs and two- outputs.

	Table (3): two-output, two-input																
Existing							BCC-I	[CRA-	I		Modified by			
							и	$_{o} \leq \varepsilon = -$	-1								
x_1	x_{2}	y_1	y_2	${oldsymbol{ heta}}^*{}_{\scriptscriptstyle BCC}$	\overline{u}_{o}	$x^{*_{1}}$	x^{*}_{2}	y_{1}^{*}	y_2^*	\hat{x}_1	\hat{x}_2	\hat{y}_1	\hat{y}_2	x^{ε}_{1}	x^{ε} 2	y^{ε}_{1}	$y^{\varepsilon}{}_{2}$
9	9	2	1	0.8642	N.F	7.7778	7.7778	2	1	10	5	4	4	12	10	6	6
12	8	3	1	0.7609	N.F	9.1308	6.0865	3	1	6	10	5	3	11.3328	9.444	5.7	5.7
7	12	2	2	0.8571	N.F	6	10	2	2	6	10	5	3	0	0	0	0
6	10	5	3	1	-2	6	10	5	3	6	10	5	3	0	0	0	0
10	5	4	4	1	-1	10	5	4	4	10	5	4	4	12.3334	7.3332	7.4	3.4
8	10	3	3	0.8750	N.F	8	10	3	3	10	5	4	4	12	10	6	6
12	10	6	6	1	$-\infty$	12	10	6	6	6.72	9.1	4.82	3.18	12	10	6	6
14	6	8	2	1	$-\infty$	14	6	8	2	10	5	4	4	12	10	6	6
12	12	1	6	1	$-\infty$	12	10	1	6	6	10	5	3	0	0	0	0
8	8	3	5	1	-1.5	8	8	3	5	10	5	4	4	0	0	0	0
98	90	37	33	-	-	92.909	82.864	37	33	80.72	74.1	44.82	35.18	71.6662	56.7772	37	33

As observed in Table (3), in this case modification has been successful at reducing the sum of inputs.

6. **Comparison and conclusions**

For more emphasize, this section, the comparison has been done between results of examples three different case in three examples. However, modified models by bounded free variable have better station at present tables into other mentioned models. Table (4) shows comparison the results for Example 1.

	Sum of inputs	Sum of outputs
	I_1	O_1
Existing DMUs	38	54
BCC-I	30.9	54
CRA-I	27.6	54
$u_o \leq \varepsilon = 0.8$ Modified by:	27.5995	55.199

Table (1): comparison for one input one output acce

Table (5) shows comparison for the results of Example2.

Table (5): two-input, one-output case								
	Sum of inputs Sum of outputs							
	I_1	I_2	O_1					
Existing DMUs	22	11.5	9					
BCC-I	22	11.5	9					
CRA-I	22	11.5	9					
$u_o \leq \varepsilon = 0$ Modified by:	19.173	10.01956	9					

	Sum o	f inputs	Sum of outputs			
	I_1	I_2	O_1	O_2		
	98	90	37	33		
Existing DMUs						
Golany.et al (1993)	97.9918	89.9972	-	-		
Beasly (2003)	88.2	81	34.0557	37.48		
BCC-I	92.91	82.86	50.52	39.48		
CRA-I						
Radial / Input oriented	80.72	74.10	44.82	35.18		
Non Radial / Input oriented	100	50	40	40		
$w_1 = w_2 = 0.5$						
Non Radial / Input oriented	63	97	47	33		
$w_1 = 1, w_2 = 0.5$						
Non Radial / Input oriented	100	50	40	40		
$w_1 = 0.5, w_2 = 1$						
$u_o \leq \varepsilon = -1$ Modified by:	72.67	56.78	37	33		

Table (6): two-input, two-output case

Table (6) compares BBC-I and radial CRA-I and modified model based on results of Table 3 and in addition in this table a comparing has been do for non radial CRA-I methods Athanassopoulos, A.D., 1995 and 1998, Roll, Y., Golany, B., 1993 Ruggiero, J., 1998, and methods which presented in Galony (1997) and Beasly (2003). All of methods consider for the data of Example 3. It can be seen that modified method implied acceptable results then others.

Acknowledgement

This paper is supported by Eastern Mediterranean University and Islamic Azad University Tabriz Branch.

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