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Maximum Likelihood Estimator for Markov Switching Autoregressive Conditional Heteroskedasticity Model

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ABSTRACT

If ARCH model changes in the parameter through a Markov-switching process then it is called Markov Switching Autoregressive Conditional Heteroskedasticity (SWARCH). In the model, conditional variance in any regime has different persistence. In this paper, we present the estimate of this model by Maximum Likelihood method.
KEYWORDS: ARCH models; Markov switching; Maximum Likelihood Estimator.

INTRODUCTION

The volatility clustering in many financial time series is implied persistence of states of high volatility. This persistence of states of high volatility leads to the rejection of standard time series; therefore Autoregressive Conditionally Heteroskedasticity (ARCH) model is proposed.

Nevertheless, if a spuriously high persistence be due to volatility clustering also the poor forecasting using ARCH model, then ARCH model will not satisfactory for studying this data. A number of researchers have suggested that the poor forecasting performance and spuriously high persistence of ARCH model might relate to structural change in the ARCH process. For these reasons, Hamilton and Susmel (1994) introduced Markov-switching ARCH (SWARCH) model. They achieved better fitting and forecasting to the New York Stock Exchange Data by SWARCH model. In this model, Markov switching process generates limited persistence.

Further applications of switching ARCH models in financial econometrics include modeling of stock market returns (Hamilton and Lin, 1996; Fornari and Mele,1997), interest rates (Gray, 1996; Ang and Bekaert, 2002), and exchange rate data (Klaasen, 2001).

The paper is structured as follows. Section 2 defined SWARCH model .sections 3 we present Maximum Likelihood Estimate of SWARCH model.

2-Model Definition

One popular approach to modeling volatility is the autoregressive conditional heteroskedasticity (ARCH) specification introduced by Engle (1982). Let $\{y_t\}$ denote a discrete time stochastic process and Ψ_{t-1} the information available at time t - 1. In this process, conditional mean of y_t given Ψ_{t-1} is:

$$E[y_t|\Psi_{t-1}] = \mu_t$$

Then, define the innovation process by $u_t = y_t - \mu_t$. In addition, the conditional variance of process given Ψ_{t-1} is:

$$var(y_t|\Psi_{t-1}) = E(u_t^2|\Psi_{t-1}) = \sigma_t^2$$

Therefore, The ARCH model of order q is given by

$$\begin{aligned} y_t &= \mu_t + u_t \\ u_t &= \sigma_t \epsilon_t \qquad , \ \epsilon_t \sim N(0,1) \\ \sigma_t^2 &= \gamma_t + \alpha_1 u_{t-1}^2 + \ldots + \alpha_q u_{t-q}^2 \end{aligned} \qquad (1)$$

With $\gamma_t = \gamma$. Where $\gamma_t > 0$, $\alpha_i \geq 0$, j =1,..q-1, and $\alpha_q > 0$. The parameter restrictions in Eq. (1) are a necessary and sufficient condition for positivity of the conditional variance.

An alternative parameterization of this model reads:

$$\begin{aligned} y_t &= \mu_t + u_t \\ u_t &= \sqrt{\gamma_t}\, h_t \epsilon_t \qquad , \ \epsilon_t \sim N(0,1) \\ h_t^2 &= 1 + \frac{\alpha_1}{\gamma_{t-1}} u_{t-1}^2 + \ldots + \frac{\alpha_q}{\gamma_{t-q}} u_{t-q}^2 \end{aligned} \qquad (2)$$

ARCH model with Markov Switching introduced by Hamilton and Susmel (1994) into parameterization (2) is:

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$$y_t = \mu_t + u_t$$
$$u_t = \sqrt{\gamma_{S_t}}\, h_t \epsilon_t \qquad , \quad \epsilon_t \sim N(0,1)$$
$$h_t^2 = 1 + \frac{\alpha_1}{\gamma_{S_{t-1}}} u_{t-1}^2 + \ldots + \frac{\alpha_q}{\gamma_{S_{t-q}}} u_{t-q}^2 \qquad (3)$$

The model is called The Markov Switching ARCH model (SWARCH). In Eq. (3) ARCH process depend on the unobserved regimes, s_t . In this model, with changes in the regime, only parameter γ_{S_t} will has changed.
It is assumed that s_t follows an ergodic M- state Markov process with an irreducible transition matrix as follows

$$P = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{N1} \\ p_{12} & p_{22} & & p_{N2} \\ \vdots & & \ddots & \vdots \\ p_{1N} & p_{2N} & \cdots & p_{NN} \end{bmatrix}$$

Where $p_{i1} + p_{i2} + \ldots + p_{iN} = 1$, for $i = 1, \ldots, N$. The row j, column i element of P is the transition probability p_{ij}. Therefore, if $s_t = i$, then $\gamma_{S_t} = \gamma_i$.
Gray (1996) the switching ARCH model has been extended by switching into all coefficients of the ARCH process.

3-Maximum Likelihood Estimate of SWARCH model

To make the analysis simple, we focus on an SWARCH model of order 1. However, an extension of the procedure to the general SWARCH of order q case would be straightforward. For the SWARCH of order 1 case, we have

$$y_t = \mu_t + u_t$$
$$u_t = \sqrt{\gamma_{S_t}}\, h_t \epsilon_t \qquad , \quad \epsilon_t \sim N(0,1)$$
$$h_t^2 = 1 + \frac{\alpha_1}{\gamma_{S_{t-1}}} u_{t-1}^2$$

Conditional variance of u_t is given by

$$E[u_t^2 | s_t, s_{t-1}, u_{t-1}] = \gamma_{S_t}\left[1 + \frac{\alpha_1}{\gamma_{S_{t-1}}} u_{t-1}^2\right]$$
$$= \sigma_t^2(s_t, s_{t-1})$$

Calculation of the density of y_t given past information (Ψ_{t-1}) is:

$$f(y_t | \Psi_{t-1}, s_t, s_{t-1}) = \frac{1}{\sqrt{2\pi}\, \sigma_t(s_t, s_{t-1})} \exp\left\{\frac{-(y_t - \mu_t)^2}{2\sigma_t^2(s_t, s_{t-1})}\right\} \qquad (4)$$

Thus, the log likelihood function is given by

$$\ln L = \sum_{t=1}^{T} \ln f(y_t | \Psi_{t-1})$$
$$= \sum_{t=1}^{T} \ln\left\{\sum_{s_t=1}^{N} \sum_{s_{t-1}=1}^{N} f(y_t, s_t, s_{t-1} | \Psi_{t-1})\right\}$$
$$= \sum_{t=1}^{T} \ln\left\{\sum_{s_t=1}^{N} \sum_{s_{t-1}=1}^{N} f(y_t | s_t, s_{t-1}, \Psi_{t-1})\, P[s_t, s_{t-1} | \Psi_{t-1}]\right\} \qquad (5)$$

And (5) can be maximized with respect to parameters of population, $p_{11}, p_{12}, \ldots, p_{NN}, \alpha_1$ $, \gamma_1, \ldots, \gamma_N$, with restriction $p_{i1} + p_{i2} + \ldots + p_{iN} = 1$ for $i = 1, \ldots, N$.
For calculation of the log likelihood function, we need S_t and S_{t-1}, which are unobserved. Therefore, with applying the repetitive filtering provided in the next section is obtainable.

3-1. Filtering

First, we calculate the steady-state probabilities $\pi = P(s_0 = i)$ for $i = 1, \ldots, N$. We obtain for two-state and first-order Markov switching as follows:
The eigenvalues of the transition matrix P for any N-state Markov chain are achieved from the solutions to $|P - \lambda I_N| = 0$. For the two-state Markov chain, the eigenvalues satisfy

$$0 = \begin{vmatrix} p_{11}-\lambda & 1-p_{22} \\ 1-p_{11} & p_{22}-\lambda \end{vmatrix}$$
$$= (\lambda-1)(\lambda+1-p_{11}-p_{22})$$

Thus, the eigenvalues for a two-state chain are given by $\lambda_1 = 1$ and $\lambda_2 = -1 + p_{11} + p_{22}$.
The eigenvector associated with λ_1 for the two-state chain turns out to be

$$\pi = \begin{bmatrix} \frac{1-p_{22}}{2-p_{11}-p_{22}} \\ \frac{1-p_{11}}{2-p_{11}-p_{22}} \end{bmatrix} \qquad (6)$$

Elements of π are steady –state or unconditional probabilities of regimes.
Then iterating in the following steps for t=1,2,…,T:
Step 1-

$$\mathrm{P}[\mathrm{s_t} = \mathrm{j}, \mathrm{s_{t-1}} = \mathrm{i}|\Psi_{t-1}] = \mathrm{P}[\mathrm{s_t} = \mathrm{j}|\mathrm{s_{t-1}} = \mathrm{i}]\ \mathrm{P}[\mathrm{s_{t-1}} = \mathrm{i}|\Psi_{t-1}]$$

Where $\mathrm{P}[\mathrm{s_t} = \mathrm{j}|\mathrm{s_{t-1}} = \mathrm{i}]$, i=1, …, N, are the transition probabilities. For t=1 is given by

$$\mathrm{P}[\mathrm{s_{t-1}} = \mathrm{i}|\Psi_{t-1}] = \mathrm{P}(\mathrm{s_0} = \mathrm{i})$$

This is the same the steady-state probability.
Step 2- Once $\mathrm{y_t}$ is observed at the end of the t-th iteration, we can update the probability terms in the following way:

$$\mathrm{P}[\mathrm{s_t} = \mathrm{j}, \mathrm{s_{t-1}} = \mathrm{i}|\Psi_t] = \frac{\mathrm{f}(\mathrm{y_t}, \mathrm{s_t} = \mathrm{j}, \mathrm{s_{t-1}} = \mathrm{i}|\Psi_{t-1})}{\mathrm{f}(\mathrm{y_t}|\Psi_{t-1})}$$

And

$$\mathrm{P}[\mathrm{s_t} = \mathrm{j}|\Psi_t] = \sum_{\mathrm{s_{t-1}}=1}^{\mathrm{N}} \mathrm{P}[\mathrm{s_t} = \mathrm{j}, \mathrm{s_{t-1}} = \mathrm{i}|\Psi_t]$$

Filtered probabilities refer to inferences about S_t conditional on information up to t i.e. Ψ_t. Then, we achieve smoothed probabilities refer to inferences about S_t conditional on all the information in the sample i.e. Ψ_T. The smoothed probabilities obtained by the smoothing algorithm provided in the next section.

3-2. Kim's Smoothing

Kim's Smoothing estimates smoothed probabilities using all the information in the sample. The Kim's smoothing algorithm (it considered by Kim 1994) as follow:

$$P[s_t = j, s_{t+1} = k|\Psi_T] = \frac{P[s_{t+1} = k|\Psi_T]P[s_t = j|\Psi_t]P[s_{t+1} = k|s_t = j]}{P[s_{t+1} = k|\Psi_t]}$$

And

$$P[s_t = j|\Psi_T] = \sum_{k=1}^{N} P[s_t = j\ , s_{t+1} = k|\Psi_T]$$

The above procedure can be iterated for t=T-1,T-2, … ,1. $P[s_t|\Psi_T]$ for t=T-1,T-2, … ,1 are estimation of the smoothed probabilities. $P[s_T|\Psi_T]$ is estimated at the last iteration of the basic filter in 3.1 section.

3- Conclusions

The ARCH model is applied for studying time series data with volatility clustering in their behavior. On the other hand, the spuriously high persistence of the volatility clustering is suggested that the use of Markov switching ARCH model. So, when allowing for a switching regime in some parameters, process has less volatility persistence.

We need to the unobserved S_t and S_{t-1} with the order of using maximum likelihood method to estimate SWARCH model. For this cause, we achieve Filtered and smoothed probabilities. Filtered probabilities are inferences about S_t conditional on information up to t. also, smoothed probabilities are inferences about S_t conditional on all the information in the sample.

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