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COHEN-MACAULAY PROPERTY OF FINITE CYCLIC SUBGROUP, l-GROUP, PI-LATTICE, AND ZARISKI SPECTRUM

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ABSTRACT

In this paper, let R be a commutative ring with an identity, $L(G)$ be the lattice of all subgroups of G,$L(R)$ be the lattice of all ideals of R, $(X,\vee,\wedge,0,1)$ be a PI-lattice, and $\Im$ be the compact open of the Zariski spectrum of R. It is shown that, if $L(R)$ is principal lattice and G is a finite cyclic group, then $R[L(G)]$ and $R[\Im]$, and $R[X]$ are Cohen-Macaulay. Finally, it show that, if $L(R)$ is principal lattice and G is a l-group, then $R[G][X_1, X_2, \ldots]$ is WB-height-unmixed.

Keywords: Distributive lattice; BCK-algebras; WB-height-unmixed; l-group; PI-lattice.

1 INTRODUCTION

A notion of "order" plays an important role in the theory of algebraic structures. Many of the key results of the theory relate important properties of algebraic structures and classes of such strutures to questions of order, e.g., the ordering of substructures, congruence relations, etc. Order also plays an important role in the computational part of the theory; for example, recursion can conveniently be defined as the least fixed point of an interative procedure. The most important kind of ordering in the general theory of algebras is a lattice ordering, which turns out to be definable by identities in terms of of the least-upper-bound (the join) and greatest-lower-bound (the meet) operations. A lattice is a poset P any pair of elements x, y have a g.l.b. or **meet** denote by $x \wedge y$, and a l.u.b. or **join** denote by $x \vee y$. We will discuss properties of Cohen-Macaulay ring. In section 2, it shows that if $L(R)$ is a principal lattice, then R is Cohen-Macaulay ring. In section 3, it is proved if G is a finite cyclic group, $R[L(G)]$ is Cohen-Macaulay. Also, In section 4, it shows that if G is a l-group, then $R[G][X_1, X_2, \ldots]$ is WB- height-unmixed. In section 5, we show that if $(X,\vee,\wedge,0,1)$ is a PI-lattice, then $R[X]$ is Cohen-Macaulay. Finally, in section 6, it indicates if $\Im$ is the compact open of the Zariski spectrum of R, then $R[\Im]$ is Cohen-Macaulay.

2 Cohen-Macaulay and principal lattice

Let L be a lattice. We say that L is complete, if every subset of L has a supremum. However, without a good notion of principal lattice, it is impossible to get very deep results, see [3]. Dilworth overcame this in [4], with a new notion of a principal element. Basically, an element E of a multiplicative lattice L is said to be meet (join-) principal if $(A \wedge (B{:}E))E = (AE) \wedge B$ (if $(BE \vee A){:}E = B \vee (A{:}E)$) for all A and B in L. A principal element is an element that is both meet-principal and join-principal or $A \wedge E = (A{:}E)E$ and $AE{:}E = A \vee (0{:}E)$, for all $A \in L$. L is called principal lattice when each of its elements is principal. Here the residual quotient of two element A, B is denoted by $A{:}B$, so $A{:}B = \vee\{X \in L | XB \leq A\}$. In this paper, it shown that, if R is Cohen-Macaulay ring and G is cyclic group, then $R[L(G)]$ is Cohen-Macaulay.

Let N be a finitely generated module over a Noetherian ring R. We say that $x \in R$ is an N-regular element, if $xg = 0$ for $g \in N$ implies $g = 0$, in the other words, if x is not a zero-divisor on N. A sequence $x_1, \cdots x_r$ of elements of the ring R, is called an N-regular sequence or simply an N-sequence if the following conditions are satisfied:

(1) x_i is an $N/(x_1, \cdots, x_{i-1})N$-regular element for $i = 1, \ldots, r$;

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(2) $N/(x_1, \dots, x_r)N \neq 0$.

Suppose $I \subseteq R$ is an ideal with $IN \neq N$. The *depth* of I on N is maximal length of an N-regular sequence in I, denoted by $depth(I, N)$. If R is a local ring with a unique maximal ideal $\mathfrak{m}$, we write $depth(\mathfrak{m})$, for $depth(\mathfrak{m}, N)$. Let R be a Noetherian local ring. A finitely generated R-module N, is a Cohen-Macaulay module, if $depth(N) = dim(N)$. If R itself is a Cohen-Macaulay module, then it is called a Cohen-Macaulay ring.

REMARK 2.1. Let S be a multiplicatively closed subset of R. For a submodule M of N, we use $S(M)$ to denote the submodule $\bigcup_{s \in S} (M :_N s)$. $Ass_R(N/S(M)) = \{P \in Ass_R(N/M): P \cap S = \emptyset\}$. Suppose that R is Noetherian and N is finitely generated. Then for any submodule M of N, $Ass_R\left(\frac{N}{M}\right) = mAss_R\left(\frac{N}{M}\right)$and N is a Cohen-Macaulay module where m is a minimal prime ideal over $Ann_R(x)$. In particular, every multiplication Noetherian ring is Cohen-Macaulay (see[11], [12]).

THEOREM 2.1. [5] If R is Cohen-Macaulay ring and P is a distributive lattice, then $R[P]$ is Cohen-Macaulay.

THEOREM 2.2. [9] Let R be a commutative ring with identity. Then $L(R)$ is a principal lattice, if and only if, R is a Noetherian multiplication ring.

3 Cohen -Macaulay property of finite cyclic group

Let $\sum$ consist of the subgroups of any group G, and let $\leq$ mean set-inclusion. Then $\sum$ is a complete lattice, with $H \wedge K = H \cap K$ (set-intersection), and $H \vee K$ the least subgroup in $\sum$ containing H and K (which is not their set-theoretical union). We now turn our attention to the lattice $L(G)$ of all subgroups of G.

Example. Let G be a group and L(G) be the set of all subgroups of G ordered by inclusion. Then it is well known that L(G) is an algebraic lattice. This lattice need not be modular.

THEOREM 3.1. Let R be a commutative ring with identity, $L(R)$ be a principal lattice and G be a finite cyclic group. Then $R[L(G)]$ is Cohen-Macaulay.

Proof. Let Z_r be a finite cyclic group of order r, with generator a. Then [2] every subgroup of Z_r is cyclic, with generator a^s for some $s|r$. Hence the lattice of positive integers, under the relation $m|n$. This shows that the lattice of all subgroups of any finite cyclic group is a distributive lattice. By Theorem 2.1, $R[L(G)]$ is Cohen-Macaulay.

4 Cohen -Macaulay property of l-group

The algebraic theory of l-groups is that of lattice generally. More generally , a group equipped with a partial order $\leq$ is called a po-group if $axb \leq ayb$ whenever $x \leq y$. An l-group is simply a po-group in which any two elements have a l.u.b and g.l.b. We begin this section by a definition from Bourbaki.

DEFINITION 4.1. A prime ideal P is an associated prime of I, if $P = I{:}\,x$ for some $x \in R$.
Remember that the height of a prime ideal P is the maximum length of the chains of prime ideals of the following form,

$$P_1 \subset P_2 \subset \cdots \subset P_k = P.$$

We will denote the height of P by $ht(P)$. An ideal I of R is said to be height-unmixed, if all the associated primes of I have equal height. That is $ht(P) = ht(Q)$, for all $P, Q \in Ass(I)$, where $Ass(I)$ denotes the set of associated primes of I. An ideal I is said to be unmixed if there are no embedded primes among the associated primes of I. That is, $P \subseteq Q \Rightarrow P = Q$, for all $P, Q \in Ass(I)$. We will say that an ideal is WB-height-unmixed, if it is height-unmixed with respect to the set of weak Bourbaki associated primes and an ideal is WB-unmixed if it is unmixed with respect to the set of weak Bourbaki associated primes. The set of weak Bourbaki associated primes of an ideal I is denoted by $Ass_f(I)$. A prime ideal P is a weak Bourbaki associated prime of the ideal I if it is a minimal ideal of the form $I{:}\,a$, for some $a \in R$.

If R is a Noetherian ring, then $R[X_1, X_2, \dots]$, the ring of polynomials in the variables $X_1, X_2, \dots$., satisfies GPIT (generalized principal ideal theorem), and it is WB-height-unmixed(see [1], [10]).

THEOREM 4.1. Let R be a commutative ring with identity, $L(R)$ be a principal lattice and G be a l-group. Then $R[G][X_1, X_2, \dots]$ is WB-height-unmixed.

Proof. In any l-group, we have $a - (a \wedge b) + b = b \vee a$ for all a, b. It suffices to show that $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ imply $x = y$. But $x = (a \wedge x) - a + (x \vee a) = (a \wedge y) - a + (y \vee a) = y$. So, G is a distributive lattice. As a result,using Theorem 2.1 and Theorem 2.2, $R[G][X_1, X_2, \dots]$ is Cohen-Macaulay.

5 Cohen -Macaulay property of PI-lattice

The study of BCK-algebras was initiated by Y. Imai and K. Iseki [6] in 1966 as a generalization of the concept of set-theoretic difference and propositional caculus. In section, relationship BCK-algebra and Cohen-Macaulat is considerd.

DEFINITION 5.1. [7] Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a BCK-algebra if it satisfies the following axioms:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCK-2) $(x * (x * y)) * y = 0$,
(BCK-3) $x * x = 0$,
(BCK-4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(BCK-5) $0 * x = 0$.

A partial ordering $\leq$ on X can be defined by $x \leq y$ if only if $x * y = 0$. A BCK-algebra X is called bounded if there exists the greatest element of X.

PROPOSITION 5.1. [8] Let X be a BCK-algebras with condition S. Then for any $x, y \in X$:

(1) $y \leq x \circ (y * x)$,
(2) $(x \circ z) * (y \circ z) \leq x * y$,
(3) $(x * y) * z = x * (y \circ z)$.

THEOREM 5.1. [6] Suppose X is a BCK-algebra with condition S. Then X is positive implicative if and only if $(X, \leq)$ is an upper semilattice with $x \vee y = x \circ y$ for any $x, y \in X$.

In this section we suppose that X is a bounded BCK-algebra, unless otherwise is stated.

DEFINITION 5.2. A lattice BCK-algebra X is called PI-lattice if it satisfies in $(z * x) * (y * x) = z * (x * y)$.

THEOREM 5.2. Let R be a commutative ring with identity, $L(R)$ be a principal lattice and $(X, \vee, \wedge, 0, 1)$ be a PI-lattice. Then $R[X]$ is Cohen-Macaulay.

Proof. Let $(X, \vee, \wedge, 0, 1))$ is a PI-lattice. Then X is a BCK-algebra with condition S and $x \circ (y * x) = x \vee y$, the converse does not hold. Also, $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$ and $x \wedge y = (x * (x * y)) \vee (y * (y * x))$ and $x * (y \vee z) = (x * y) \vee (x * z)$ and $x * (y \vee z) = (x * y) \wedge (x * z)$ and $(x \vee y) * z = (x * z) \vee (y * z)$. So, X is a distributive and modular lattice and applying Theorem 2.1,$R[X]$ is Cohen-Macaulay.

6 Cohen -Macaulay property of Zariski spectrum

THEOREM 6.1. Let R be commutative ring with an identity. If $L(R)$ is a principal lattice and $\mathfrak{I}$ be the compact open of the Zariski spectrum of R. Then $R[\mathfrak{I}]$ is Cohen-Macaulay.

Proof. The set of all prime ideals of a ring R has a natural topology with basic open

$$D(a) = \{p | a \notin p\}$$

We clearly have

$$D(a) \cap D(b) = D(ab), D(0) = \emptyset$$

The space of all topology, in general non Hausdorff. Though we cannot describe the points of this space effectively in general, we can describe the topology of the space effectively. The compact open of the spectrum are of the form

$$D(a_1, \ldots, a_n) = D(a_1) \cup \ldots \cup D(a_n)$$

The compact open form a distributive lattice. By applying Theorem 2.1, $R[X]$ is Cohen-Macaulay.

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