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Cubic Bézier Constrained Curve Interpolation

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ABSTRACT

Bézier function is one of the substantial polynomial and fundamental tool for interpolation because it is easy to compute and implement. In this paper, we develop cubic Bézier constrained curve interpolation. The end points of cubic Bézier function are left for user's choice. Simple constraints are derived on two middle points of cubic Bézier function constrained by a circle, an ellipse and straight line with point of intersection. Furthermore, the cubic Bézier function represents the S-shaped and C-shaped curves. The developed scheme is tested through different numerical examples and found to be computationally economical and visually pleasant.

KEY WORDS*:* Computer Aided Geometric Design, Cubic Bézier function, Interpolation, End points, S-shaped curve, C-shaped curve.

1. INTRODUCTION

Curves and surfaces design is an important topic of CAGD (Computer Aided Geometric Design) and computer graphics. CAGD is concerned with algorithms for the design of smooth curves and surfaces and has efficient mathematical representation. Bézier is one of the imperative polynomial and important tool for interpolation. The Bézier interpolating curve always lies within the convex hull and never wanders from the control polygon. Bézier polynomial has several applications in the fields of engineering, science and technology such as highway or railway rout designing, networks, Computer aided design system, animation, robotics, communications and many other disciplines because it is easy to compute and also stable [4,9,11]. The significance of Bézier polynomials in diverse areas, namely electronics or engineering is well known [6]. The parametric and non parametric representation of curves and surfaces especially in polynomial form is most suitable for design, as the planer curves cannot deal with infinite slopes and are axis dependent too.

Many authors have studied numerous kinds of Spline for curve and surface design, shape preservation [2, 7]. Abbas, et al [1], developed quadratic and cubic Bézier interpolations constrained by a line. The author derived simple conditions on the middle points of quadratic and cubic Bézier function to be constrained by a line. Abbas [2], developed a C^1 piecewise rational cubic function with shape parameters to preserve the shape of constrained data. Simple data dependent conditions on shape parameter were derived to preserve the shape of data lying above the straight line. Brodlie, et al [3] constructed modified quadratic Shepard method which interpolates a scattered data of any dimension to preserve the positivity. The authors inserted extra knots in the interval in such a way that the desired shape of data was preserved. Meek, et al [10], constructed a rational cubic for interpolating the given set of ordered points lying on one side of a polyline. Goodman, et al [5], developed two schemes of interpolating data to preserve the shape of data lying on one side of the straight line by non parametric rational cubic function. Firstly, they preserved the shape of data lying above the straight line by scaling the weights by some scale factor. Secondly, they preserved the shape of data by inserting new interpolation point.

Jeok, Ong [8] investigated C^1 monotonicity and G^1 constrained to lie on the same side of given constraint line using cubic Bézier-like function. Hussain, et al [7], developed C^1 piecewise rational cubic function in most general form to visualize the constrained data in the view of constrained curve that is lying above the straight line.

In this paper, we construct ordinary cubic Bézier function which is constrained by general circle, general ellipse, straight line and circle, straight line and an ellipse with point of intersection. Simple conditions are imposed on the two middle points of cubic Bézier function to be constrained by a straight line, circle and an ellipse with point of intersection. The cubic Bézier function has many advantages as compared to rational cubic function with shape parameter. Due to non rational form of the function, it is easy to compute and implement for two dimensional data. There is no need to insert additional knots where the function loses its shape. Likewise no constraint interval length is required as in rational cubic interpolations. The developed method is computationally economical, time saving and produces pleasing graphical results.

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The remaining part of paper is arranged as follows: The cubic Bézier function is discussed in section 2. The cubic Bézier interpolation constrained by a general circle is developed in section 3. The cubic Bézier interpolation constrained by circle and straight line is given in section 4. In section 5, the cubic Bézier interpolation constrained by a general ellipse is developed. In section 6, the cubic Bézier interpolation constrained by an ellipse and straight line is discussed. Finally, Conclusion of the work and future road map is given in section 7. Numerical examples are given to support the competency of the developed constrained interpolation.

2. CUBIC BÉZIER FUNCTION

Let $\left(P_i=(x_i,f_i), i=0,1,2,3\right)$ be the four control points of cubic Bézier function and $f_i, i=0,1,2,3$ are Bézier points of the function. The cubic Bézier function is defined as:

$$r(t)=\sum_{i=0}^{3} B_i^3(t) f_i, i=0,1,2,3 \qquad 0\le t\le 1 \tag{1}$$

where,

$$\begin{cases} B_0^3(t)=(1-t)^3, B_1^3(t)=t(1-t)^2 \\ B_2^3(t)=t^2(1-t), B_3^3(t)=t^3 \end{cases} \tag{2}$$

are Bernstein cubic basis polynomials, where $t=(x-x_0)/a, a=x_3-x_0$.
The end point conditions are defined as,

$$r(x_0)=f_0\ ,\ r(x_3)=f_3 \tag{3}$$

3. CUBIC BÉZIER CURVE INTERPOLATION CONSTRAINED BY A GENERAL CIRCLE WITH POINT OF INTERSECTION

In this section, the cubic Bézier function (1) is constrained by a general circle when the end points are given. Simple conditions are derived for two middle Bézier points that guarantee the curve to be constrained by the general circle $(x-h)^2+(y-k)^2=r^2$ with centre $C(h,k)$ and radius r. Also the cubic Bézier function represents the form of S-shaped and C-shaped curves. So, it is required to impose suitable conditions on the two middle Bézier points by some mathematical treatment for the curve to be constrained by the general circle with point of intersection as follows:
Let $C(h,k)$ and r be the centre and radius of circle respectively, then the equation of circle is:

$$y=k+\sqrt{r^2-(x-h)^2} \tag{4}$$

The necessary condition on given end Bézier points is,

$$f_0, f_3 \ge y \tag{5}$$

Let $p(p_x,p_y)$ or $p(p_x,r(p_x))$ be the intersection point of the cubic Bézier function (1) and circle in equation (4), such that,

$$p_x=h+rCos\theta, \quad p_y=k+rSin\theta \qquad 0<\theta<\pi$$

Where θ is an anti clockwise angle to control the curve to be constrained by a circle as shown in Fig.1.
The cubic Bézier function is constrained by a general circle if,

$$r'(x)=y' \tag{6}$$

where prime represents the derivative w.r.t "x". After simple calculations and using equation (6), the value of cubic Bézier function point f_1 is:

$$f_1=\frac{\left(a^2(p_x-x_0)\left((h-p_x)(3h-2p_x-x_0)-3(r^2+C(k-2f_0))\right)-a^3A-6aB\right)}{3C(p_x-x_0)(a-p_x+x_0)} \tag{7}$$

where,

$A=\left(2h^2+p_x(p_x+x_0)-h(3p_x+x_0)-2(r^2+C(k-f_0))\right)$,

$B=C(p_x-x_0)^2 f_0+C(p_x-x_0)^3(2f_0+f_3)$,

$C=\sqrt{r^2-(h-p_x)^2}$

For the value of second point f_2, we solve the following equation,

$$r(x)=y \tag{8}$$

That gives,

$$f_2=\frac{-1}{9aC(p_x-x_0)^2(a-p_x+x_0)}\left(3C(a-3p_x+3x_0)D+3(p_x-x_0)(a-p_x+x_0)E\right) \tag{9}$$

where,

$$D=\left(a^3(k+C-f_0)+3a^2(p_x-x_0)f_0-3a(p_x-x_0)^2f_0+(p_x-x_0)^3(f_0-f_3)\right),$$

$$E=\left(a^3(p_x-h)-3C(a-p_x+x_0)^2f_0+3C(p_x-x_0)^2f_3\right),$$

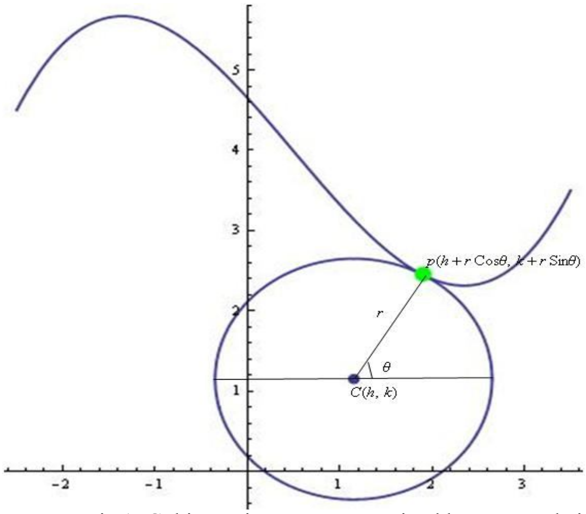


Fig.1: Cubic Bézier curve constrained by a general circle

THEOREM 3.1: Let $r(t)$ be the cubic Bézier function (1) and y be the general circle (4) with centre $C(h,k)$ and radius r. Then $r(t)$ is constrained by the general circle with necessary conditions defined in equation (5) if equations (7) and (9) are satisfied.

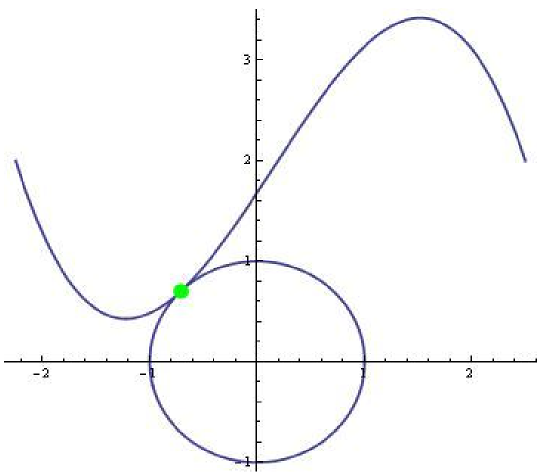
PROOF

The proof is obvious from the information given above.

EXAMPLE 3.1

Consider two circles centred at $C(0,0)$ and $C(0.75,0.75)$ with radius $r=1$. The cubic Bézier function (1) is constrained by a general circle with point of intersection and also represents an S-shaped and C-shaped curves respectively, if $(x_0,f_0)=(-2.25,2)$, $(x_3,f_3)=(2.5,2)$, $\theta=\frac{3\pi}{4}$ and $(x_0,f_0)=(-1.25,4.5)$, $(x_3,f_3)=(5.5,3.5)$, $\theta=\frac{\pi}{3}$ as shown in Fig.2 (a) and (b).

a)



b)

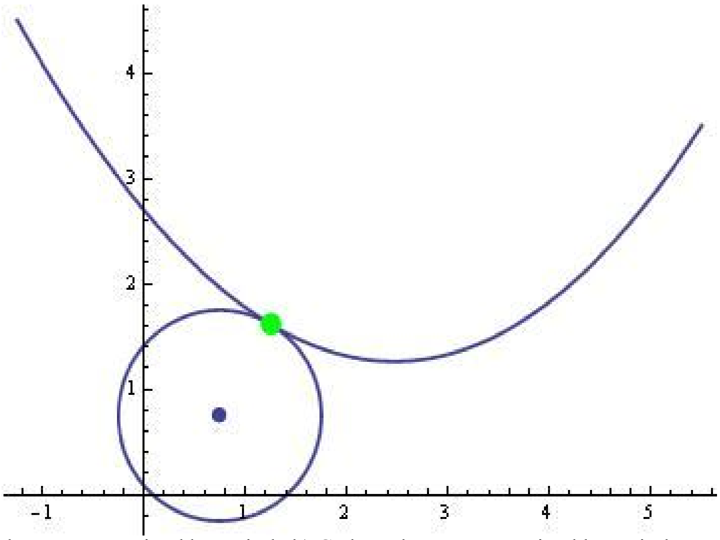


Fig.2: a) S-shaped curve constrained by a circle b) C-shaped curve constrained by a circle

4. CUBIC BÉZIER CURVE INTERPOLATION CONSTRAINED BY A CIRCLE AND STRAIGHT LINE WITH POINT OF INTERSECTION

In this section, the cubic Bézier function (1) is constrained by a circle and any straight line when the end Bézier points are given. The curve is enforced to be constrained by unit circle $x^2 + y^2 = (r^2 = 1)$ with centre $C(0,0)$ and any straight line $y = mx + c$ by applying simple conditions on two middle Bézier points. Also it is represented in the form of S-shaped and C-shaped curves.

Let the straight line be:

$$y = mx + c \tag{10}$$

Where 'm' is the slope and 'c' is the y-intercept of line defined as,

$$\begin{cases} m = -Cot\theta \\ c = 1/Sin\theta \end{cases} \tag{11}$$

The necessary condition on given end Bézier points is:

$$f_0, f_3 \geq y \tag{12}$$

Let $p(p_x, p_y)$ or $p(p_x, r(p_x))$ or $p(p_x, y(p_x))$ be the intersection point of the cubic Bézier function, circle and straight line as shown in Fig.3 such that,

$$p_x = Cos\theta, \quad p_y = Sin\theta \qquad 0 < \theta < \pi \tag{13}$$

Where θ is the anti clockwise angle to control the curve.

The cubic Bézier function is constrained by a circle and straight line if,

$$r'(x) = y' \tag{14}$$

where prime represents the derivative w.r.t "x". From equation (14), the value of cubic Bézier function point f_1 is:

$$f_1 = \frac{a^3(2c + m(p_x - x_0) - 2f_0) - a^2(p_x - x_0)(3c + 2mp_x + mx_0 - 6f_0) - 6a(p_x - x_0)^2\left(f_0 - (p_x - x_0)(2f_0 + f_3)\right)}{3(p_x - x_0)(a - p_x + x_0)^2} \tag{15}$$

For the value of second point f_2, we solve

$$r(x) = y \tag{16}$$

That gives,

$$f_2 = \frac{-a^3(c + mx_0 - f_0) + a^2(p_x - x_0)(3c + 2mp_x + mx_0 - 3f_0) + 3a(p_x - x_0)^2\left(f_0 - (p_x - x_0)(f_0 + 2f_3)\right)}{3(p_x - x_0)^2(a - p_x + x_0)} \tag{17}$$

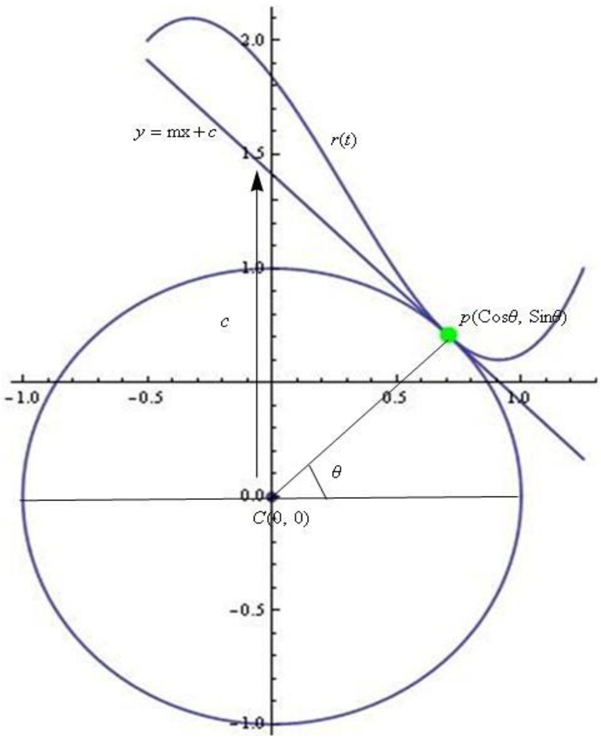


Fig.3: Cubic Bézier function constrained by a circle and straight line.

THEOREM 4.1 Let $r(t)$ be the cubic Bézier function (1) and $y = mx + c$ be any straight line as defined in equation (10). The cubic Bézier function $r(t)$ is constrained by a circle and straight line with point of intersection along with necessary conditions defined in equation (12) if conditions derived in equations (15) and (17) are satisfied.

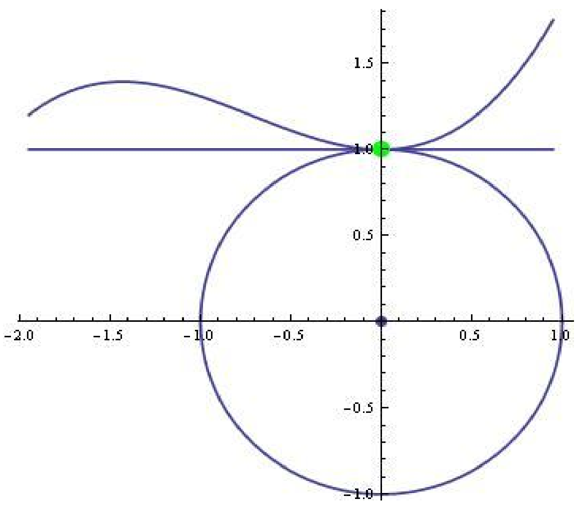
PROOF

The proof is straightforward.

EXAMPLE4.1

Let $C(0,0)$ be the centre of both circles with radius $r = 1$. The cubic Bézier function (1) is constrained by a circle and straight line with point of intersection and for $(x_0, f_0) = (-1.95, 1.2)$, $(x_3, f_3) = (0.95, 1.75)$, $\theta = \frac{\pi}{2}$ and $(x_0, f_0) = (-2.5, 2.2)$, $(x_3, f_3) = (1.5, 3)$, $\theta = \frac{5\pi}{9}$,it represents an S-shaped and C-shaped curves constrained by the line respectively as shown in Fig.4 (a) and (b).

a)



b)

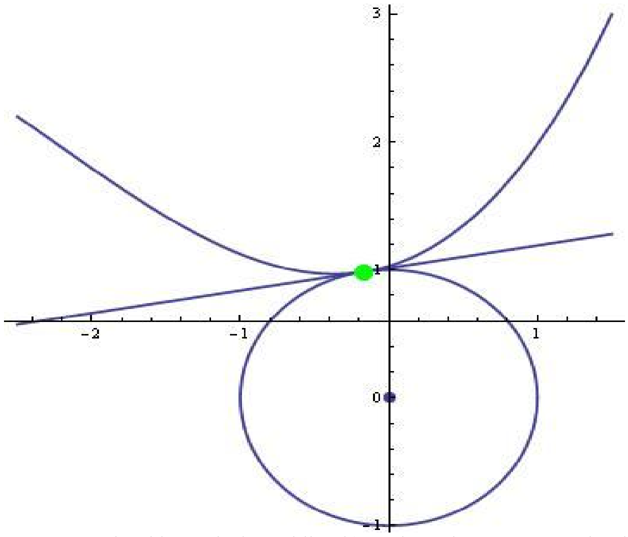


Fig.4: a) S-shaped curve constrained by a circle and line b) C-shaped curve constrained by a circle and line

5. CUBIC BÉZIER CURVE INTERPOLATION CONSTRAINED BY GENERAL ELLIPSE WITH POINT OF INTERSECTION

In this section, the cubic Bézier function (1) is constrained by the general ellipse when the end Bézier points are given. Conditions applied on two middle Bézier points assure that the curve would be constrained by general ellipse $\frac{(x-h)^2}{u^2}+\frac{(y-k)^2}{v^2}=1$ with centre $C(h,k)$, $'u'$ as semi major axis, and $'v'$ as semi minor axis. Also the cubic Bézier function is represented in the form of S-shaped and C-shaped curves.

Let the general equation of ellipse be:

$$y = k + \frac{v}{u}\sqrt{u^2-(x-h)^2} \tag{18}$$

The necessary condition on given end Bézier points is,

$$f_0, f_3 \geq y \tag{19}$$

Let $p(p_x, r(p_x))$ or $p(p_x, p_y)$ be the intersection point of the cubic Bézier function (1), and general ellipse (18) as shown in Fig.5 such that,

$$p_x = h + u\,Cos\theta, \quad p_y = k + v\,Sin\theta \qquad 0<\theta<\pi \tag{20}$$

where θ is the anti clockwise angle to control the curve.

The cubic Bézier function is constrained by a general ellipse if,

$$r'(x) = y' \tag{21}$$

where prime represent the derivative w.r.t $"x"$. Using equation (21), the value of cubic Bézier function ordinate f_1 is:

$$f_1 = \frac{va^2\left(u^2(2a-3p_x+3x_0)+(h-p_x)A\right)+uB\left(2a^3(k-f_0)-3a^2(p_x-x_0)(k-2f_0)-(p_x-x_0)^2\left(6af_0-(p_x-x_0)(2f_0+f_3)\right)\right)}{3uB(p_x-x_0)(a-p_x+x_0)^2} \tag{22}$$

Where,

$A = (3h-2p_x-x_0)(p_x-x_0)+a(-2h+p_x+x_0),$

$B = \sqrt{(u+h-p_x)(u-h+p_x)}$

For the value of second point f_2, we solve the equation:

$$r(x) = y \tag{23}$$

That produces,

$$f_2 = \frac{-va^2\left(u^2(a-3p_x+3x_0)+(h-p_x)C\right)+uB\left(-a^3(k-y_0)+3a^2(p_x-x_0)(k-y_0)+(p_x-x_0)^2\left(3ay_0-(p_x-x_0)(y_0+2y_3)\right)\right)}{3uB(p_x-x_0)^2(a-p_x+x_0)} \tag{24}$$

where,
$C = (3h - 2p_x - x_0)(p_x - x_0) + a(-h + x_0),$
$B = \sqrt{(u + h - p_x)(u - h + p_x)}$

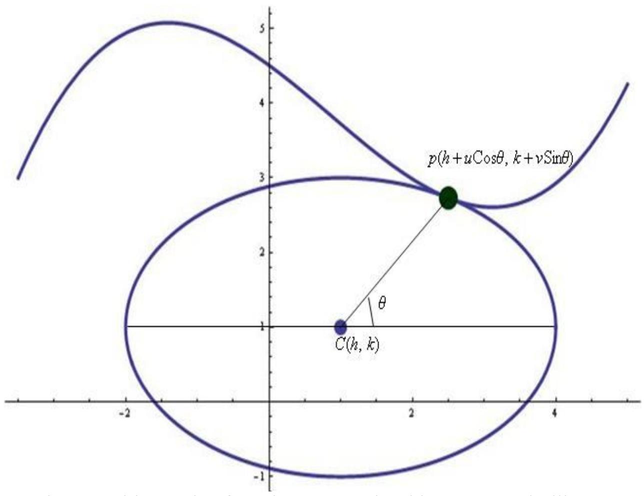


Fig.5: Cubic Bézier function constrained by a general ellipse.

THEOREM 5.1 Let $r(t)$ be the cubic Bézier function and y be the general ellipse as defined in equations (1) and (18), respectively. The cubic Bézier function $r(t)$ is constrained by general ellipse with point of intersection along with necessary conditions derived in equation (19) if the conditions on two middle points given in equations (22) and (24) are satisfied.

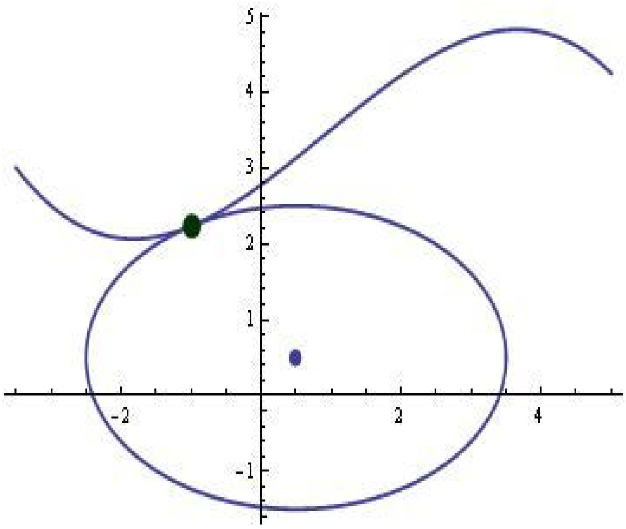
PROOF

The result follows immediately from the above discussion.

EXAMPLE 5.1

Let $C(0.5, 0.5)$ be the centre of ellipse of Fig.6 (a) with $u = 3$, $v = 2$ and $C(1.5, 1.5)$ be the centre of ellipse of Fig.6 (b) where $u = 2, v = 3$. The cubic Bézier function (1) is constrained by the general ellipse with point of intersection. Also for $(x_0, f_0) = (-3.5, 3)$, $(x_3, f_3) = (5, 4.25)$, $\theta = \frac{2\pi}{3}$ and $(x_0, f_0) = (-3, 5)$, $(x_3, f_3) = (4, 6.25)$, $\theta = \frac{5\pi}{9}$, respectively, it represents an S-shaped and C-shaped curves as seen in Fig.6(a) and (b).

a)



b)

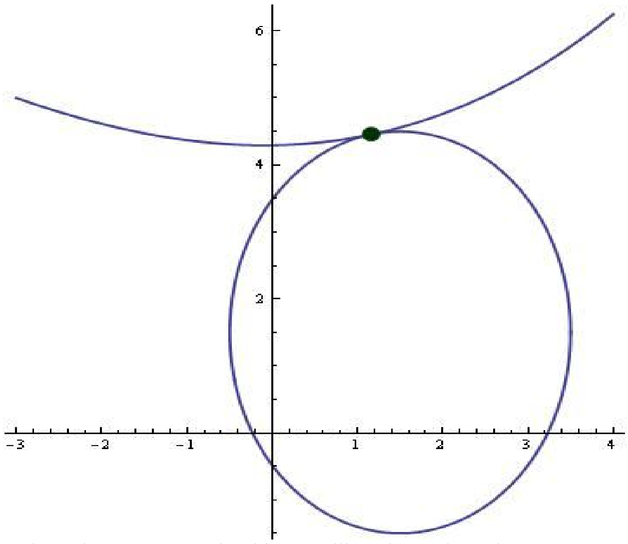


Fig.6: a) S-shaped curve constrained by an ellipse b) C-shaped curve constrained by an ellipse

6. CUBIC BÉZIER CURVE INTERPOLATION CONSTRAINED BY AN ELLIPSE AND STRAIGHT LINE WITH POINT OF INTERSECTION

In this section, the cubic Bézier function (1) is constrained by an ellipse and straight line when the end Bézier points are given. The curve promises to be constrained by an ellipse $x^2/u^2 + y^2/v^2 = 1$ with centre $C(0,0)$, $'u'$ semi major axis, $'v'$ semi minor axis and any straight line $y = mx + c$ due to specially constructed conditions on two middle Bézier points. Also the cubic Bézier function is presented in the form of S-shaped and C-shaped curves.

Let the straight line be:

$$y = mx + c \tag{25}$$

where $'m'$ is the slope and $'c'$ is the y intercept of line defined as:

$$\begin{cases} m = -\dfrac{v}{uCot\theta} \\ c = vSin\theta - muCos\theta \end{cases} \tag{26}$$

The necessary condition on given end Bézier points to be constrained by an ellipse and straight line is:

$$f_0, f_3 \geq y \tag{27}$$

Let $p(p_x, r(p_x))$ or $p(p_x, p_y)$ or $p(p_x, y(p_x))$ be the intersection point of the cubic Bézier function (1), an ellipse and straight line (25) as shown in Fig.7 such that,

$$p_x = uCos\theta, \quad p_y = vSin\theta \qquad 0 < \theta < \pi \tag{28}$$

Where θ be the anti clockwise angle to control the curve,

The cubic Bézier function is constrained by an ellipse and straight line if,

$$r'(x) = y' \tag{29}$$

where prime represents the derivative w.r.t $"x"$. After simple calculations, the value of cubic Bézier function ordinate f_1 is:

$$f_1 = \frac{a^3(2c + m(p_x + x_0) - 2f_0) - a^2(p_x - x_0)(3c + 2mp_x + mx_0 - 6f_0) - 6a(p_x - x_0)^2\left(f_0 - (p_x - x_0)(2f_0 + f_3)\right)}{3(p_x - x_0)(a - p_x + x_0)^2} \tag{30}$$

For the value of second point f_2, we solve the equation,

$$r(x) = y \tag{31}$$

And get

$$f_2 = \frac{-a^3(c + mx_0 - f_0) + a^2(p_x - x_0)(3c + 2mp_x + mx_0 - 3f_0) + 3a(p_x - x_0)^2\left(f_0 - (p_x - x_0)(f_0 + 2f_3)\right)}{3(p_x - x_0)^2(a - p_x + x_0)} \tag{32}$$

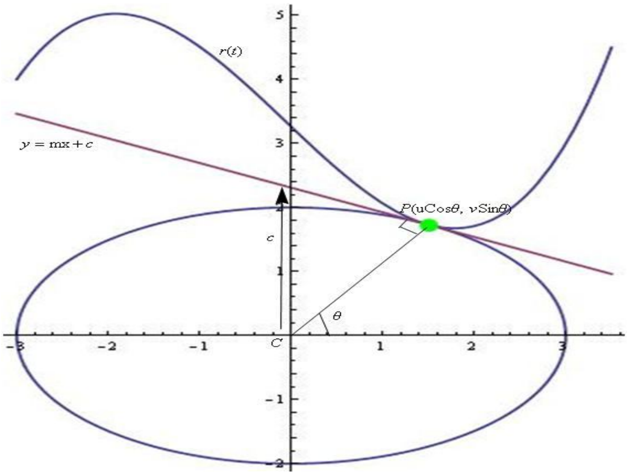


Fig.7: Cubic Bézier function constrained by an ellipse and straight line.

THEOREM 6.1 Let $r(t)$ be the cubic Bézier function (1) and $y = mx + c$ be any straight line as given in equation (25). The cubic Bézier function $r(t)$ is constrained by an ellipse and straight line with point of intersection along with necessary conditions defined in equation (27) if the sufficient conditions on middle points of function defined in equations (30) and (32) are satisfied.

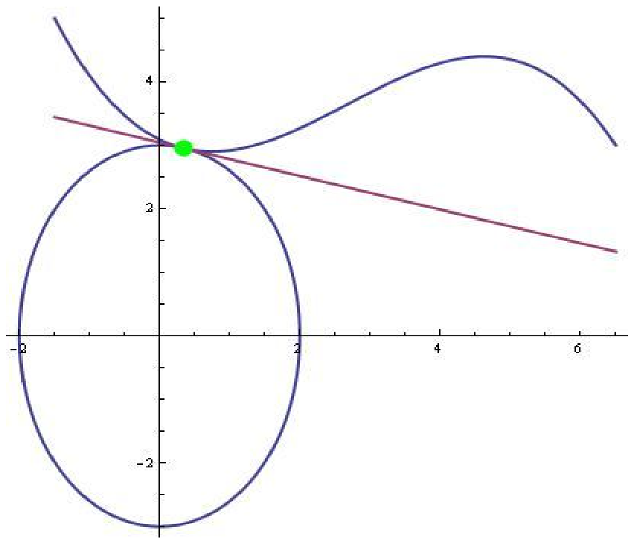
PROOF

The preceding computations lead to the proof.

EXAMPLE 6.1

Let $C(0,0)$ be the centre of both ellipse with semi major axes $u = 3$, semi minor axes $v = 2$ and $u = 2, v = 3$ respectively. The cubic Bézier function (1) is constrained by an ellipse and straight line with point intersection and also represented an S-shaped and C-shaped curves respectively, if $(x_0, f_0) = (-1.5, 5)$, $(x_3, f_3) = (6.5, 3)$, $\theta = \frac{4\pi}{9}$ and $(x_0, f_0) = (-4.75, 3)$, $(x_3, f_3) = (1, 5.5)$, $\theta = \frac{2\pi}{3}$ as shown in Fig.8(a) and (b).

a)



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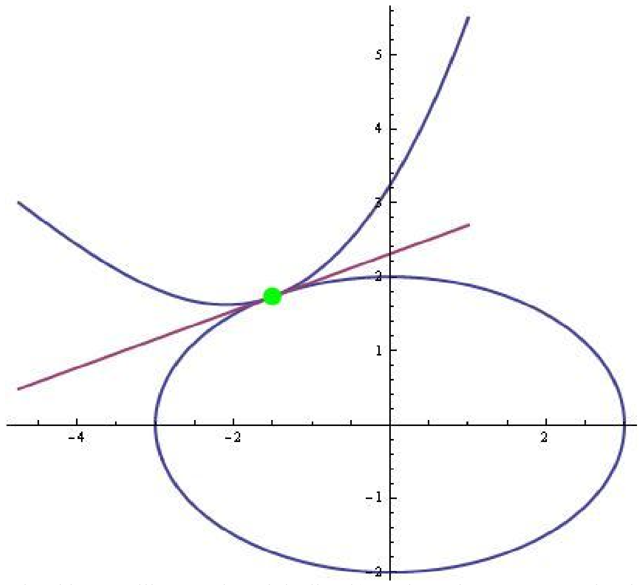

Fig.8: a) S-shaped curve constrained by an ellipse and straight line b) C-shaped curve constrained by an ellipse and straight line

7. CONCLUDING REMARKS

In this paper, we have constructed ordinary cubic Bézier function which is constrained by general circle, general ellipse, straight line and circle, straight line and an ellipse with point of intersection. Simple conditions are derived for the two middle points of cubic Bézier function. They insure the user that the curve is constrained by a straight line, circle and an ellipse with point of intersection. The cubic Bézier function has advantage over the rational cubic function with shape parameters. The non rational form of the function makes it simple to compute without any constraint interval length and easy to apply to two dimensional data. In contrast to [3], no additional knots are inserted between any two knots to attain the desired shape. The developed method is computationally economical, time saving and visually pleasing. In future we would construct Quartic and Quintic Bézier function curve and surface schemes.

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