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Fixed and Periodic Point Results for $T-$ Quasi-Contractions in a Partially Ordered Metric Space

Vahid Parvaneh *

Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran.

ABSTRACT

In this paper, we study the existence of the fixed point for $T-$quasi-contractive type mappings in the setup of partially ordered spaces. We also introduce $T-$ generalized weakly quasi-contractive mappings and present necessary conditions to obtain fixed point for such mappings in ordered spaces. As an application of our results, periodic points of $T-$ quasi-contractions is obtained. We also provide examples to illustrate the results presented herein.

KEYWORDS: Fixed Point, Quasi-Contraction, Periodic Point, Ordered Metric Space, Complete Metric Space.

1. INTRODUCTION AND PRELIMINARIES

Let (X,d) be a metric space. A self map f on X is said to be a Banach contraction mapping, if there exists a number $k \in [0,1)$ such that

$$d(fx, fy) \leq kd(x, y)$$

for all $x, y \in X$.

If f is a Banach contraction mapping on a complete metric space X, then by Banach contraction principle, f has a unique fixed point, that is, there exists one and only one $x \in X$ such that $f(x) = x$. Banach contraction principle has several applications in different branches of mathematics.

As a generalization of Banach contraction mapping, the notion of $T-$contraction mapping has been introduced by Beiranvand et al. [3].

Let (X,d) be a metric space. A map $f: X \to X$ is called a quasi-contraction if for some constant $\alpha \in [0,1)$ and for every $x, y \in X$,

$$d(fx, fy) \leq \alpha \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}. \quad (1)$$

This concept was introduced and studied by Lj. Ciric [4], in 1974. A result of Ciric shows that every quasi-contraction f, defined on a complete metric space has an unique fixed point and recently, in [9] and [10] some fixed point theorems for quasi-contractive mappings in cone metric spaces have been proved.

Definition 1.1 *A mapping $f: X \to X$ is said to be a T-quasi-contraction if*

$$d(Tfx, Tfy) \leq \alpha \max\{d(Tx, Ty), d(Tx, Tfx), d(Ty, Tfy), d(Tx, Tfy), d(Ty, Tfx)\},$$

for all $x, y \in X$, where $\alpha \in [0,1)$.

If $T = I$ (the identity mapping on X), then the above definition reduces to the definition of quasi-contraction mapping.

Definition 1.2 *Let (X,d) be a metric space. A mapping $f: X \to X$ is said to be sequentially convergent (subsequentially convergent) if for a sequence $\{x_n\}$ in X for which $\{fx_n\}$ is convergent, $\{x_n\}$ also is convergent ($\{x_n\}$ has a convergent subsequence).*

***Corresponding Author:** Vahid Parvaneh, Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran.
E-mail: vahid.parvaneh@kiau.ac.ir

Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [14], and then by Nieto and Lopez [12].

In this paper, we establish some fixed point theorems for quasi-contractive type mappings in a partially ordered complete metric space.

2 MAIN RESULTS

Throughout this paper, let $(X,\leq)$ be a partially ordered set, $F(f)=\{x\in X: fx=x\}$ be the fixed point set of f, $(LF)_f=\{x\in X: x\leq fx\}$ be the lower fixed point set of f, and

$$M(Tx,Ty)=\max\{d(Tx,Ty),d(Tx,Tfx),d(Ty,Tfy),d(Tx,Tfy),d(Ty,Tfx)\}.$$

We start with the following result. In fact, we show that under some appropriate conditions, every T-quasi-contraction f defined on a complete partially ordered metric space X with $\alpha\in[0,\frac{1}{2})$ has a fixed point in X.

Theorem 2.1 *Let $(X,\leq,d)$ be a complete partially ordered metric space and $T:X\to X$ be an injective, continuous subsequentially convergent mapping. If $f:X\to X$ be a nondecreasing map such that for every elements $x,y\in X$ with $x\leq y$,*

$$d(Tfx,Tfy)\leq\alpha\max\{d(Tx,Ty),d(Tx,Tfx),d(Ty,Tfy),\\ d(Tx,Tfy),d(Ty,Tfx)\}, \tag{2}$$

where $\alpha\in[0,\frac{1}{2})$, then $F(f)\neq\phi$ provided that there exists an $x_0\in (LF)_f$, and one of the following two conditions is satisfied:

(a) f is continuous self map on X;

(b) for any nondecreasing sequence $\{x_n\}$ in X such that $x_n\to z$ as $n\to\infty$, it follows that $x_n\leq z$ for all $n\in\mathbb{N}$.

Moreover, f has a unique fixed point iff the fixed points of f are comparable.

Proof. Since $x_0\in (LF)_f$ and f is nondecreasing, therefore $f^nx_0\leq f^{n+1}x_0$ for each $n\in\mathbb{N}$. Define a sequence $\{x_n\}$ in X with $x_n=f^nx_0$ and so $x_{n+1}=fx_n$ for all $n\in\mathbb{N}$. If there exists a positive integer n such that $x_n=x_{n+1}$, then $f^nx_0=f^{n+1}x_0=ff^nx_0$ implies that f^nx_0 is a fixed point of f. Assume that, $x_n\neq x_{n+1}$ for every positive integer n. Since $x_{n-1}\leq x_n$, therefore by replacing x by x_{n-1} and y by x_n in 2, we have

$$\begin{aligned}
d(Tx_n,Tx_{n+1}) &= d(Tfx_{n-1},Tfx_n)\\
&\leq\alpha\max\{d(Tx_{n-1},Tx_n),d(Tx_{n-1},Tfx_{n-1}),d(Tx_n,Tfx_n),\\
&\qquad d(Tx_{n-1},Tfx_n),d(Tx_n,Tfx_{n-1})\}\\
&=\alpha\max\{d(Tx_{n-1},Tx_n),d(Tx_{n-1},Tx_n),d(Tx_n,Tx_{n+1}),\\
&\qquad d(Tx_{n-1},Tx_{n+1}),d(Tx_n,Tx_n)\}\\
&\leq\alpha\max\{d(Tx_{n-1},Tx_n),d(Tx_n,Tx_{n+1}),d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\}\\
&=\alpha[d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})],
\end{aligned}$$

which further implies

$$d(Tx_{n+1},Tx_n)\leq hd(Tx_n,Tx_{n-1}),$$

where $h=\dfrac{\alpha}{1-\alpha}$. Obviously, $0\le h<1$. Repeating the above process, we get,

$$d(Tx_{n+1},Tx_n)\le hd(Tx_n,Tx_{n-1})\le \ldots \le h^n d(Tx_1,Tx_0),$$

for all $n\ge 1$, and so for $m>n$, we have

$$\begin{aligned}
d(Tx_n,Tx_m) &\le d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})+\ldots+d(Tx_{m-1},Tx_m)\\
&\le h^n d(Tx_0,Tx_1)+h^{n+1}d(Tx_0,Tx_1)+\ldots+h^{m-1}d(Tx_0,Tx_1)\\
&= h^n(1+h+\ldots+h^{m-n-1})d(Tx_0,Tx_1)\\
&\le \frac{h^n}{1-h}d(Tx_0,Tx_1).
\end{aligned}$$

It follows that $\lim_{n\to\infty} d(Tx_n,Tx_m)=0$. Since X is complete, there exists an element $z\in X$ such that $\lim_{n\to\infty} Tf^n x_0 = z.$

As T is subsequentially convergent, so we have $\lim_{i\to\infty} f^{n_i}x_0=u$ for some u in X, where $\{f^{n_i}x_0\}$ is a subsequence of $\{f^n x_0\}$. Since T is continuous, $\lim_{i\to\infty} Tf^{n_i}x_0=Tu$ which by uniqueness of limit, implies that $Tu=z.$

If f is continuous selfmap on X, then $Tfu=Tu$, therefore we have $fu=u$. If f is not continuous, then by the given assumption $x_{n_i}=f^{n_i}x_0\le u$ for all $n\in\mathbb{N}$, it follows that

$$\begin{aligned}
d(Tu,Tfu) &\le d(Tfx_{n_i},Tfu)+d(Tfx_{n_i},Tu)\\
&\le \alpha\max\{d(Tx_{n_i},Tu),d(Tx_{n_i},Tfx_{n_i}),d(Tu,Tfu),\\
&\qquad d(Tx_{n_i},Tfu),d(Tu,Tfx_{n_i})\}+d(Tfx_{n_i},Tu)\\
&= \alpha\max\{d(Tx_{n_i},Tu),d(Tx_{n_i},Tx_{n_i+1}),d(Tu,Tfu),\\
&\qquad d(Tx_{n_i},Tfu),d(Tu,Tx_{n_i+1})\}+d(Tx_{n_i+1},Tu)\\
&\le \alpha\max\{d(Tx_{n_i},Tu),d(Tx_{n_i},Tx_{n_i+1}),d(Tu,Tfu),\\
&\qquad d(Tx_{n_i},Tu)+d(Tu,Tfu),d(Tu,Tx_{n_i+1})\}+d(Tx_{n_i+1},Tu),
\end{aligned}$$

which, on taking the limit as $i\to\infty$, implies that

$$d(Tu,Tfu)\le \alpha d(Tu,Tfu),$$

and hence $d(Tu,Tfu)=0$ or equivalently $Tu=Tfu$. So $u=fu$.

Suppose that fixed points of f are comparable. Let w be another fixed point of f such that $w\ne u$. With out any loss of generality, we assume that $u\le w$. Using (2), we obtain that

$$\begin{aligned}
d(Tu,Tw) &= d(Tfu,Tfw)\\
&\le \alpha\max\{d(Tu,Tw),d(Tu,Tfu),d(Tw,Tfw),d(Tu,Tfw),d(Tw,Tfu)\}\\
&= \alpha\max\{d(Tu,Tw),d(Tu,Tu),d(Tw,Tw),d(Tu,Tw),d(Tw,Tu)\}\\
&\le \alpha d(Tu,Tw),
\end{aligned}$$

and hence $d(Tu,Tw)=0$ which further implies that $u=w$ as T is injective.

Remark 2.2 *The conclusion of Theorem 2.1 holds if we replace the subsequential convergence assumption of f by sequential convergence assumption.*

Example 2.3 *Let* $X=[0,1]$ *be endowed with the usual ordering and let* d *be the usual metric on* X. *Let* $T, f : X \to X$ *be defined by* $Tx = x^2$ *and* $fx = x/2$. *For any* $x, y \in X$ *with* $x \le y$,

$$\begin{aligned} d(Tfx,Tfy) &= \frac{1}{4}(y^2-x^2) \\ &\le \frac{1}{4}(y^2-\frac{x^2}{4}) \\ &= \frac{1}{4}\max\{(y^2-x^2), \frac{3}{4}x^2, \frac{3}{4}y^2, \left|x^2-\frac{y^2}{4}\right|, (y^2-\frac{x^2}{4})\} \\ &= \alpha \max\{d(Tx,Ty), d(Tx,Tfx), d(Ty,Tfy), d(Tx,Tfy), d(Ty,Tfx)\}. \end{aligned}$$

Thus (2) is satisfied with $\alpha=\frac{1}{4}$. Obviously, f is continuous and nondecreasing and T is injective, continuous and sequentially convergent. Thus all conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique fixed point of f.

If $T = I_X$ (the identity mapping on X) in Theorem 2.1, then we obtain the following result.

Theorem 2.4 *Let* $(X, \le, d)$ *be an ordered complete metric space and let* $f : X \to X$ *be a nondecreasing map such that for every elements* $x, y \in X$ *with* $x \le y$,

$$d(fx, fy) \le \alpha \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}, \tag{3}$$

where $\alpha \in [0, \frac{1}{2})$. If there exists $x_0 \in X$ with $x_0 \le fx_0$, and one of the following two conditions is satisfied:

(a) f is a continuous self map on X;

(b) for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to z$ as $n \to \infty$, it follows $x_n \le z$ for all $n \in \mathbb{N}$, then $F(f) \ne \phi$.

Moreover, f has an unique fixed point provided that the fixed points of f are comparable.

Example 2.5 *Let* $X=[0,1]$ *be endowed with usual order and usual metric and* $f : X \to X$ be defined by $fx = \frac{x^2}{4}$.

Clearly, f is continuous and nondecreasing. Let $x, y \in X$ with $x \le y$. Then

$$\begin{aligned} d(fx,fy) &= \frac{1}{4}(y^2-x^2) \\ &\le \frac{1}{4}(y-\frac{x^2}{4}) \\ &= \frac{1}{4}\max\{(y-x), x-\frac{1}{4}x^2, y-\frac{1}{4}y^2, \left|x-\frac{1}{4}y^2\right|, y-\frac{1}{4}x^2\} \\ &= \alpha \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\} \\ &= \alpha d(y,fx). \end{aligned}$$

Therefore, (3) is satisfied with $\alpha = \frac{1}{4} < \frac{1}{2}$. Thus all the conditions of Theorem 2.6 are satisfied. Moreover, 0 is the unique fixed point of f.

Theorem 2.6 *Let* $(X,\leq)$ *be a partially ordered set such that there exists a complete metric* d *on* X *and* $T:X\to X$ *be an injective, continuous subsequentially convergent mapping. Let* $f:X\to X$ *be a nondecreasing map such that for every elements* $x,y\in X$ *with* $x\leq y$,

$$d(Tfx,Tfy)\quad\leq\frac{1}{2}M(Tx,Ty)-\varphi(M(Tx,Ty)),\tag{4}$$

and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function such that $\varphi(t)>0$ for all $t\in(0,\infty)$ and $\varphi(0)=0$. Then $F(f)\neq\phi$ provided that there exists an $x_0\in\ (LF)_f$, and one of the following two conditions is satisfied:

(a) f is continuous self map on X ;

(b) for any nondecreasing sequence $\{x_n\}$ in X such that $x_n\to z$ as $n\to\infty$, it follows that $x_n\leq z$ for all $n\in\mathbb{N}$.

Moreover, f has an unique fixed point provided that the fixed points of f are comparable.

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. If there exists a positive integer n such that $x_n=x_{n+1}$, then x_n is a fixed point of f. Assume that, $x_n\neq x_{n+1}$, for every positive integer n. Since $x_{n-1}\leq x_n$, therefore by replacing x by x_{n-1} and y by x_n in (2), we have

$$\begin{aligned}d(Tx_n,Tx_{n+1})&=d(Tfx_{n-1},Tfx_n)\\&\leq\frac{1}{2}M(Tx_{n-1},Tx_n)-\varphi(M(Tx_{n-1},Tx_n)),\end{aligned}$$

where

$$\begin{aligned}M(Tx_{n-1},Tx_n)&=\max\{d(Tx_{n-1},Tx_n),d(Tx_{n-1},Tfx_{n-1}),d(Tx_n,Tfx_n),\\&\qquad d(Tx_{n-1},Tfx_n),d(Tx_n,Tfx_{n-1})\}\\&=\max\{d(Tx_{n-1},Tx_n),d(Tx_{n-1},Tx_n),d(Tx_n,Tx_{n+1}),\\&\qquad d(Tx_{n-1},Tx_{n+1}),d(Tx_n,Tx_n)\}\\&\leq\max\{d(Tx_{n-1},Tx_n),d(Tx_n,Tx_{n+1}),\\&\qquad d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\},\end{aligned}$$

Assume that $d(Tx_n,Tx_{n-1})\leq d(Tx_{n+1},Tx_n)$, for some positive integer n. Then we have $M(Tx_{n-1},Tx_n)\leq 2d(Tx_n,Tx_{n+1})$, and so

$$d(Tx_{n+1},Tx_n)\leq d(Tx_{n+1},Tx_n)-\varphi(M(Tx_{n-1},Tx_n)),$$

therefore,

$$\varphi(M(Tx_{n-1},Tx_n))\leq 0,$$

that is, $M(Tx_{n-1},Tx_n)=0$, which implies that $Tx_n=Tx_{n+1}$ or that $x_n=x_{n+1}$, contradicting our assumption that $x_n\neq x_{n+1}$, for each n. Therefore, $d(Tx_{n+1},Tx_n)<d(Tx_n,Tx_{n-1})$, for all $n\geq 0$ and so $d(Tx_{n+1},Tx_n)$ is a monotone decreasing sequence of non-negative real numbers. Hence, there exists an $r\geq 0$ such that $\lim_n d(Tx_{n+1},Tx_n)=r$.

From the above facts we have for all $n\geq 0$,

$$d(Tx_n,Tx_{n+1})\leq M(Tx_{n-1},Tx_n)\leq d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n),$$

so we have, $r\leq\lim_n M(Tx_{n-1},Tx_n)=l\leq 2r.$

Taking the upper limit as $n\to\infty$ in the above inequality, and since φ is l.s.c., we have

$$r \le r - \varphi(l),$$

so we have $r = l = 0$. Hence

$$\lim_{n} d(Tx_{n+1}, Tx_n) = 0. \tag{5}$$

Next we show that $\{Tx_n\}$ is a Cauchy sequence. If not, then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$d(Tx_{m(k)}, Tx_{n(k)}) \ge \varepsilon. \tag{6}$$

This means that,

$$d(Tx_{m(k)}, Tx_{n(k)-1}) < \varepsilon. \tag{7}$$

From 6 and triangle inequality

$$\begin{aligned} \varepsilon \le d(Tx_{m(k)}, Tx_{n(k)}) &\le d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)}) \\ &< \varepsilon + d(Tx_{n(k)-1}, Tx_{n(k)}). \end{aligned}$$

Letting $k \to \infty$ and using 7 we can conclude that

$$\lim_{k\to\infty} d(Tx_{m(k)}, Tx_{n(k)}) = \varepsilon. \tag{8}$$

Moreover, from

$$| d(Tx_{n(k)+1}, Tx_{m(k)}) - d(Tx_{n(k)}, Tx_{m(k)}) | \le d(Tx_{n(k)+1}, Tx_{n(k)})$$

and

$$| d(Tx_{m(k)+1}, Tx_{n(k)}) - d(Tx_{m(k)}, Tx_{n(k)}) | \le d(Tx_{m(k)+1}, Tx_{m(k)})$$

and

$$| d(Tx_{m(k)+1}, Tx_{n(k)+1}) - d(Tx_{m(k)}, Tx_{n(k)+1}) | \le d(Tx_{m(k)+1}, Tx_{m(k)})$$

and using 5 and 8 we get

$$\begin{aligned} \lim_{k\to\infty} d(Tx_{m(k)+1}, Tx_{n(k)}) &= \lim_{k\to\infty} d(Tx_{m(k)}, Tx_{n(k)+1}) \\ &= \lim_{k\to\infty} d(Tx_{m(k)+1}, Tx_{n(k)+1}) = \varepsilon. \end{aligned} \tag{9}$$

As $\{x_n\}$ is nondecreasing and $n(k) > m(k)$, from 2

$$\begin{aligned} & d(Tx_{n(k)+1}, Tx_{m(k)+1}) = d(Tfx_{n(k)}, Tfx_{m(k)}) \\ & \le \frac{1}{2} M(Tx_{n(k)}, Tx_{m(k)}) - \varphi(M(Tx_{n(k)}, Tx_{m(k)})) \\ & \le \frac{1}{2} \max\{d(Tx_{n(k)}, Tx_{m(k)}), d(Tx_{n(k)}, Tfx_{n(k)}), d(Tx_{m(k)}, Tfx_{m(k)}), \\ & \quad d(Tx_{n(k)}, Tfx_{m(k)}), d(Tx_{m(k)}, Tfx_{n(k)})\} \\ & - \varphi(\max\{d(Tx_{n(k)}, Tx_{m(k)}), d(Tx_{m(k)}, Tfx_{m(k)})\}) \\ & = \frac{1}{2} \max\{d(Tx_{n(k)}, Tx_{m(k)}), d(Tx_{n(k)}, Tx_{n(k)+1}), d(Tx_{m(k)}, Tx_{m(k)+1}), \\ & \quad d(Tx_{n(k)}, Tx_{m(k)+1}), d(Tx_{m(k)}, Tx_{n(k)+1})\} \end{aligned}$$

$$-\varphi(\max\{d(Tx_{n(k)},Tx_{m(k)}),d(Tx_{m(k)},Tx_{m(k)+1})\}),$$

where

$$\begin{aligned}M(Tx_{n(k)},Tx_{m(k)}) &= \max\{d(Tx_{n(k)},Tx_{m(k)}),d(Tx_{n(k)},Tfx_{n(k)}),d(Tx_{m(k)},Tfx_{m(k)}),\\ &\quad d(Tx_{n(k)},Tfx_{m(k)}),d(Tx_{m(k)},Tfx_{n(k)})\}\\ &= \max\{d(Tx_{n(k)},Tx_{m(k)}),d(Tx_{n(k)},Tx_{n(k)+1}),d(Tx_{m(k)},Tx_{m(k)+1}),\\ &\quad d(Tx_{n(k)},Tx_{m(k)+1}),d(Tx_{m(k)},Tx_{n(k)+1})\}.\end{aligned}$$

Making $k \to \infty$ and taking into account 9, we have

$$\varepsilon \le \frac{1}{2}(\varepsilon) - \varphi(\varepsilon)$$

and from this inequality $\varphi(\varepsilon) = 0$. By our assumption about φ, we have $\varepsilon = 0$ which is a contradiction. So, $\{Tx_n\}$ is a Cauchy sequence.

Since X is complete, there exists an element $z \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = z$. As T is subsequentially convergent, so we have $\lim_{i\to\infty} f^{n_i} x_0 = u$ for some u in X, where $\{f^{n_i} x_0\}$ is a subsequence of $\{f^n x_0\}$. Since T is continuous, $\lim_{i\to\infty} Tf^{n_i} x_0 = Tu$ which by uniqueness of limit, implies that $Tu = z$.

If f is continuous selfmap on X, then $Tfu = Tu$, therefore we have $fu = u$. If f is not continuous, then by the given assumption $x_{n_i} = f^{n_i} x_0 \le u$ for all $n \in \mathrm{N}$, and it follows that

$$d(Tfx_{n_i},Tfu) = d(Tx_{n_i+1},Tfu) \le \frac{1}{2}M(Tx_{n_i},Tu) - \varphi(M(Tx_{n_i},Tu))$$

where

$$\begin{aligned}M(Tx_{n_i},Tu) &= \max\{d(Tx_{n_i},Tu),d(Tx_{n_i},Tfx_{n_i}),d(Tu,Tfu),\\ &\quad d(Tx_{n_i},Tfu),d(Tu,Tfx_{n_i})\}\\ &= \max\{d(Tx_{n_i},Tu),d(Tx_{n_i},Tx_{n_i+1}),d(Tu,Tfu),\\ &\quad d(Tx_{n_i},Tfu),d(Tu,Tx_{n_i+1})\},\end{aligned}$$

which, on taking the limit as $i \to \infty$, implies that

$$d(Tu,Tfu) \le \frac{1}{2}d(Tu,Tfu) - \varphi(d(Tu,Tfu)),$$

and hence $d(Tu,Tfu) = 0$ or equivalently $Tu = Tfu$. So $u = fu$.

3 Periodic point results

Clearly, a fixed point of f is also a fixed point of f^n, for every $n \in \mathrm{N}$, that is, $F(f) \subset F(f^n)$. However, the converse is false. For example, the mapping $f:\mathrm{R} \to \mathrm{R}$, defined by $fx = \frac{1}{2} - x$ has an unique fixed point $\frac{1}{4}$, but every $x \in \mathrm{R}$, is a fixed point of f^2. If $F(f) = F(f^n)$ for every $n \in \mathrm{N}$, then f is said to have property P. For more details, we refer to [16] and references mentioned therein.

Recently, the study of Periodic points for contraction mappings has been considered by many authors, for instance, every quasi-contraction $f: X \to X$ with the constant $\alpha \in [0, \frac{1}{2})$, where X is a cone metric space, has the property P ([10], Theorem 3.1.) and, if (X, d) be a cone metric space, and T -Hardy-Rogers contraction $f: X \to X$ satisfies some appropriate conditions, then f has property P ([6], Corollary 3.3.)

Definition 3.1 *[1] Let $(X, \leq)$ be a partially ordered set. A mapping f is called dominating on X if $x \leq fx$ for each x in X.*

Example 3.2 *[1] Let $X = [0,1]$ be endowed with usual ordering. Let $f: X \to X$ be defined by $fx = x^{\frac{1}{3}}$, then $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Thus f is a dominating map.*

Example 3.3 *[1] Let $X = [0, \infty)$ be endowed with usual ordering. Let $f: X \to X$ be defined by $fx = \sqrt[n]{x}$ for $x \in [0,1)$ and $fx = x^n$ for $x \in [1, \infty)$, for any $n \in \mathbb{N}$, then for all $x \in X$, $x \leq fx$, that is f is a dominating map.*

We have the following result:

Theorem 3.4 *Let $(X, \leq, d)$ be a partially ordered complete metric space and $T: X \to X$ be an injective mapping. Let $f: X \to X$ is a nondecreasing mapping such that for all $x \in X$ with $x \leq fx$, we have*

$$d(Tfx, Tf^2x) \leq \lambda d(Tx, Tfx), \tag{10}$$

where $\lambda \in [0,1)$. Then f has the property P provided that $F(f)$ is nonempty and f is dominating on $F(f^n)$.

Proof. Let $u \in F(f^n)$ for some $n > 1$. Now we show that $u = fu$. Since f is dominating on $F(f^n)$, therefore $u \leq fu$ which implies that $f^{n-1}u \leq f^n u$ as f is nondecreasing. Using (10), we obtain that

$$\begin{aligned} d(Tu, Tfu) &= d(Tff^{n-1}u, Tf^2 f^{n-1}u) \\ &\leq \lambda d(Tf^{n-1}u, Tf^n u) = \lambda d(Tff^{n-2}u, Tf^2 f^{n-2}u). \end{aligned}$$

Repeating the above process, we get

$$d(Tu, Tfu) \leq \lambda^n d(Tu, Tfu),$$

which on taking the limit as $n \to \infty$, implies that $d(Tu, Tfu) = 0$ or equivalently $Tu = Tfu$. So $u \in F(f)$.

Theorem 3.5 *Let X, T and f be as in Theorem 2.1. If f is dominating on X, then f satisfies property P.*

Proof. From Theorem 2.1, $F(f) \neq \emptyset$. We shall prove that (10) is satisfied for all $x \leq fx$. Indeed, f is dominating so that $x \leq fx$. Also, $fx \leq f^2x$, as f is nondecreasing. Using (2), we have

$$\begin{aligned} & d(Tfx, Tf^2x) = d(Tfx, Tffx) \\ & \leq \alpha \max\{d(Tx, Tfx), d(Tx, Tfx), d(Tfx, Tf^2x), \\ & \quad d(Tx, Tf^2x), d(Tfx, Tfx)\} \\ & \leq \alpha \max\{d(Tx, Tfx), d(Tfx, Tf^2x), d(Tx, Tfx) + d(Tfx, Tf^2x)\} \\ & = \alpha d(Tx, Tfx) + d(Tfx, Tf^2x)], \end{aligned}$$

that is,

$$d(Tfx, Tf^2x) \leq \lambda d(Tx, Tfx),$$

where $\lambda = \dfrac{\alpha}{1-\alpha}$. Obviously, $\lambda \in [0,1)$. By Theorem 3.4, f has property P .

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