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A Fixed Point Approach to the Non-Archimedean Random Stability of Pexider Quadratic Functional Equation with Involution

M. M. Pourpasha
Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran, Iran

ABSTRACT

In this paper, we apply the fixed point method to investigate the Hyers-Ulam stability of the Pexider functional equation

$$f(x+y)+g(x+\sigma(y))=h(x)+k(y), \forall x,y\in X,$$

in the setting of non-Archimedean Random normed spaces, where $\sigma : X \to X$ is an involution.

KEYWORDS: Non-Archimedean , Random space, Fixed Point, quadratic equation, Pexider type equation, Hyers-Ulam-Rassias stability.

1. INTRODUCTION

The stability problem of functional equations originated from a question of Ulam (1940), concerning the stability of group homomorphisms.

The stability concept that was introduced by Rassias' theorem (1978) provided a large influence to a number of mathematicians to develop the notion of what is known today with the term Hyers-Ulam-Rassias stability of the linear mapping. Since then, the stability of several functional equations has been extensively investigated by several mathematicians.

Recently, stability of functional equations has been investigated by many mathematicians. They have many applications in information theory, physics, economic theory and social and behavior sciences. A Hyers-Ulam stability theorem for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1}$$

was proved by Skof (1983) for functions $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa (1984) noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group.

In 1992, Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation (1). Recently Jung (2000), Jung and Sahoo (2001) investigated the Hyers-Ulam-Rassias stability of the functional equation of Pexider type

$$f_1(x+y)+f_2(x-y)=f_3(x)+f_4(y). \tag{2}$$

They have many applications in Information Theory, Physics, Economic Theory and Social and Behavior Sciences.

Let X and Y be real vector spaces. If an additive function $\sigma : X \to X$ satisfies $\sigma(x+y)=\sigma(x)+\sigma(y)$ and $\sigma(\sigma(x))=x$ for all $x,y\in X$, then σ is called an involution of X see [?, ?]. For a given involution $\sigma : X \to X$, the functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(y), \forall x,y\in X \tag{3}$$

is called the quadratic functional equation with involution. A function $f : X \to Y$ is a solution of (3) if and only if there exists an additive function $A : X \to Y$ and a biadditive symmetric function $B : X\times X \to Y$ such that

$$(A(\sigma(x)))=A(x), (B(\sigma(x),y))=-B(\sigma(x),y)$$

and

$$f(x)=B(x,y)+A(x)$$

***Corresponding Author:** M. M. Pourpasha, Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran, Iran. E-mail: pourpasha@eml.cc Tel: +98-911-1941811; Fax: +98 191 2220536.

for all $x \in X$.

Indeed, if we set $\sigma(x) = x$ in (3), where $I : X \to X$ denotes the identity function, then (3) reduces to the additive functional equation

$$f(x+y) = f(x) + f(y), \forall x, y \in X. \tag{4}$$

On the other hand, if $\sigma(x) = -x$ in (3), then (3) is transformed into the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \forall x, y \in X. \tag{5}$$

Recently, Bouikhalene et al (2007). have proved the Hyers-Ulam-Rassias stability of the quadratic functional equation with involution (2).

The theory of random normed spaces is important as a generalization of deterministic result of linear normed spaces and the study of random operator equations. The random normed spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, random normed spaces and fuzzy normed spaces have been recently studied in Alsina (1987), Mirmostafaee, Mirzavaziri and Moslehian (2008), Mihet and Radu (2008), Mihet, Saadati and Vaezpour (2009), Baktash et al. (2008) and Saadati et al. (2009).

In the present paper, we will apply the fixed point method to prove the Hyers-Ulam-Rassias stability of the functional equation (3) in Pexider type

$$f(x+y) + g(x+\sigma(y)) = 2h(x) + 2k(y), \tag{6}$$

in the setting of non-Archimedean normed spaces.

2 Non-Archimedean random normed space

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of non-Archimedean random normed spaces (non-Archimedean RN-spaces) as in [16, 25, 26].

Throughout this paper, Δ^+ is the space of all probability distribution functions, i.e., the space of all mappings

$$F : R \cup \{-\infty, +\infty\} \to [0,1]$$

such that F is left-continuous and non-decreasing on R, $F(0) = 0$, and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, i.e.,

$$l^- f(x) = \lim_{t \to x^-} f(x).$$

The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(x) \le G(x)$ for all x in R. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0 = \begin{cases} 0, & \text{if} x \le 0 \\ 1, & \text{if} x > 0 \end{cases}$$

A triangular norm (t-norm) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$, which is commutative, associative, and monotone, and has 1 as the unit element. Basic examples are the Lukasiewicz t-norm T_L,

$$T_L(a,b) = \max(a+b-1, 0) \ \forall a, b \in [0,1],$$

and the t-norms T_P, T_M, and T_D,

$$T_p(a,b) := a \times b,$$

$$T_M(a,b) := \min(a,b),$$

$$T_D(a,b) := \begin{cases} \min(a,b), & \max(a,b) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

A t-norm T can be extended by associativity in a unique way to an n-ary operation for $(x_1,...,x_n) \in (0,1]^n$. The value $T\ (x_1,...,x_n)$ is defined by

$$T_{i=1}^{0} x_i = 1, \qquad T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1,...,x_n).$$

T can also be extended to a countable operation for any sequence $(x_n)_{n\in N}$ in $[0,1]$. Moreover

$$T_{i=1}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{n} x_i.$$

Lemma 2.1 *[25]For $T \geq T_L$, the following implication holds:*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

By a non-Archimedean field, we mean a field K equipped with a function (valuation) $|.|$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$, and $|r+s| \leq max(|r|,|s|)$ for all $r,s \in K$. Clearly, $|1|=|-1|=1$ and $|n| \leq 1$ for all $n \in N$. The function $|.|$ is called the trivial valuation if $|r|=1$, $\forall r \in K$, $r \neq 0$, and $|0|=0$. Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|.|$. A function $\|.\| : X \to 0,\infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

- $\|x\| = 0$ if and only if $x = 0$,
- For any $r \in K$ and $x \in X$, $\|rx\| = |r|\|x\|$,
- The strong triangle inequality (ultrametric)

$$\|x+y\| \leq \max(x,y)\ (x,y \in X)$$

is satisfied.

Then, $(X,\|.\|)$ is called a non-Archimedean normed space due to the fact that

$$\|x_n - x_m\| \leq \max(\|x_{j+1} - x_j\| : m \leq j \leq n-1)\ (x,y \in X)$$

in which $n > m$, the sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in anon-Archimedean normed space. In a complete non-Archimedean normed space, every Cauchy sequence is convergent.

Definition 2.2 *A non-Archimedean RN-space is a triple (X,μ,T), where X is a linear space over a non-Archimedean field K, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:*

NA-RN1: $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

NA-RN2: $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X,\ t > 0$ and $\alpha \neq 0$,

NA-RN3: $\mu_{x+y}(\max(t,s)) \geq T(\mu_x(t), \mu_y(s))$, for all $x,y \in X$ and $t,s \geq 0$.

It is easy to see that if (NA-RN3) holds, then

RN3: $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$, for all $x,y \in X$ and $t > 0$.

Example 2.3 As a classical example, if $(X,\|.\|)$ is a non-Archimedean normed linear space, then the triple (X,N,T_M) is a non-Archimedean RN-space, where

$$\mu_x(t)=\begin{cases}0, & t\le\|x\|,\\ 0, & t>\|x\|.\end{cases}$$

Example 2.4 *Let* $(X,\|\cdot\|)$ *be a non-Archimedean normed linear space. Define*

$$\mu_x(t)=\frac{t}{t+\|x\|},$$

for all $x\in X$ and $t>0$. Then, (X,μ,T_M) is a non-Archimedean RN-space.

Definition 2.5 *Let* (X,μ,T) *be a non-Archimedean RN-space and* $\{x_n\}$ *be a sequence in* X*. Then,* $\{x_n\}$ *is said to be convergent if there exists* $x\in X$ *such that*

$$\lim_{n\to\infty}\mu_{x_n-x}(t)=1$$

for all $t>0$, where x is the limit of the sequence $\{x_n\}$.

• A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon>0$ and $t>0$, there exists n_0 such that for all $n\ge n_0$ and all $p>0$, we have

$$\mu_{x_{n+p}-x_n}(t)>1-\varepsilon.$$

Theorem 2.6 *([16]) Let* (X,N,T_M) *ibe a non-Archimedean RN-space. Then,*

$$\mu_{x_{n+p}-x_n}(t)\ge\min(\mu_{x_{n+j+1}-x_{n+j}}(t):j=0,1,2,...,p-1).$$

Therefore, the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon>0$ and $t>0$, there exists n_0 such that for all $n\ge n_0$, we have

$$\mu_{x_{n+1}-x_n}(t)>1-\varepsilon.$$

If each Cauchy sequence is convergent, then the random norm is said to be complete, and the non-Archimedean RN-space is called a non-Archimedean random Banach space.

Here we recall an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al (1997).

Theorem 2.7 *Let* (X,d) *be a generalized complete metric space. Assume that* $J:X\to X$ *is a strictly contractive operator with the Lipschitz constant* $0<L<1$*. If there exists a nonnegative integer* k *such that* $d(J^{k+1}x,J^kx)<\infty$ *for some* $x\in X$ *, then the followings are true:*

• The sequence $\{J^nx\}$ converges to a fixed point x^* of J,

• x^* is the unique fixed point of J in

$$X^*=\{y\in X:d(J^kx,x)<\infty\}.$$

• If $y\in X^*$, then

$$d(y,x^*)\le\frac{1}{1-L}d(Jy,y).$$

3 Random Hyers-Ulam stability in even cases

In this section, we investigate the stability of the Pexider quadratic functional equation with involution

$$f(x+y)+g(x+\sigma(y))=2h(x)+2k(y) \tag{7}$$

where f, g, h, k even mappings from X to Y, and $f(0)=g(0)=h(0)=k(0)=0$, by using the fixed point approach.

Next, we define a random approximately additive quadratic mapping. Let Ψ be a distribution function on $X \times X \times [0,\infty)$ such that $\Psi(x, y, z, .\ )$ is nondecreasing, and

$$\Psi(cx, cy, t) \geq \Psi(x, y, \frac{t}{|c|})$$

where $x \in X$ and $c \neq 0$.

Definition 3.1 *Mappings $f, g, h, k : X \to Y$ is said to be Ψ-approximately pexider quadratic functional equation with involution if*

$$\mu_{f(x+y)+g(x+\sigma(y))-2h(x)-2k(y)}(t) \geq \Psi(x, y, t), \tag{8}$$

where $x, y \in X$, $t > 0$, $\sigma : X \to X$ is an involution of X and $f(0) = g(0) = h(0) = k(0) = 0$.

Theorem 3.2 *Let K be a non-Archimedean field, X be a vector space over K, and (Y, μ, T_M) be a non-Archimedean random Banach space over K. Let $f, g, h, k : X \to Y$ be even Ψ-approximately Pexider quadratic functional equation with involution. If for some $\alpha \in R$, $0 < \alpha < 4$, such that*

$$\Psi(2x, 2y, t) \geq \Psi(x, y, \frac{t}{\alpha}),$$

$$\Psi(x + \sigma(x), y + \sigma(y), t) \geq \Psi(x, y, \frac{t}{\alpha}). \tag{9}$$

in which $x \in X$ and $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_M\{M(x,0,2t), M'(x, x, \frac{4-\alpha}{4}t)\},$$

$$\mu_{g(x)-Q(x)}(t) \geq T_M\{M(x,0,2t), M'(x, x, \frac{4-\alpha}{4}t), \Psi(x, x, 2t), \Psi(x, -x, 2t)\},$$

$$\mu_{h(x)-Q(x)}(t) \geq M'_{\varphi(x,x)}(\frac{4-\alpha}{4}t),$$

$$\mu_{k(x)-Q(x)}(t) \geq T_M\{\Psi(0, d, 2t), \Psi(d, 0, 2t), M'(x, x, \frac{4-\alpha}{4}t)\}, \tag{10}$$

for all $x, y \in X$ and $t > 0$ in which

$$M(x, y, t) = T_M\{\Psi(x, y, t), \Psi(0, y, t), \Psi(y, 0, t), \Psi(\frac{x+\sigma(y)}{2}, \frac{x+\sigma(y)}{2}, t), \Psi(\frac{x+\sigma(y)}{2}, -\frac{x+\sigma(y)}{2}, t)\},$$

$$M'(x, y, t) = T_M\{M(x, y, t), M(x + y, 0, 2t), M(x + \sigma(y), 0, 2t)\}.$$

Moreover,

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}[h(2^n x) + (2^n - 1)h(2^{n-1}x + 2^{n-1}\sigma(x))].$$

Proof. Letting $y = 0$ in (8), , we obtain

$$\mu_{f(x)+g(x)-2h(x)}(t) \geq \Psi(x, 0, t). \tag{11}$$

Similarly, for every $y \in X$, we can put $x = 0$ in (8) to obtain

$$\mu_{f(y)+g(\sigma(y))-2k(y)}(t) \geq \Psi(0, y, t). \tag{12}$$

Since $\sigma : X \to X$ is an involution, we replace y by $x + \sigma(x)$ in (12), then

$$\mu_{f(d)+g(d)-2k(d)}(t) \geq \Psi(0, x+\sigma(x), t). \tag{13}$$

Also, we can replace y by x and replace y by $-x$ in (8) to get

$$\mu_{f(2x)+g(x+\sigma(x))-2h(x)-2k(x)}(t) \geq \Psi(x, x, t), \tag{14}$$

$$\mu_{g(x-\sigma(x))-2h(x)-2k(-x)}(t) \geq \Psi(x, -x, t). \tag{15}$$

Since $\sigma : X \to X$ is an involution, we replace x by $x-\sigma(x)$ in (14), then

$$\mu_{f(2(x-\sigma(x)))-2h(x-\sigma(x))-2k(x-\sigma(x))}(t) \geq \Psi(x-\sigma(x), x-\sigma(x), t). \tag{16}$$

Also, replace x by $x-\sigma(x)$ in (15)

$$\mu_{g(2(x-\sigma(x)))-2h(x-\sigma(x))-2k(x-\sigma(x))}(t) \geq \Psi(x-\sigma(x), -(x-\sigma(x)), t). \tag{17}$$

In view of (11) and (13), we see that

$$\mu_{2(h(x+\sigma(x))-k(x+\sigma(x)))}(t) \geq T_M\{\Psi(0, x+\sigma(x), t), \Psi(x+\sigma(x), 0, t)\}. \tag{18}$$

It follows from (16) and (17) that

$$\mu_{f(2(x-\sigma(x)))-g(2(x-\sigma(x)))}(t) \geq T_M\{\Psi(x-\sigma(x), x-\sigma(x), t), \Psi(x-\sigma(x), -(x-\sigma(x)), t)\}. \tag{19}$$

By using (8), (18) and (19), we have

$$\begin{aligned} &\mu_{f(x+y)+f(x+\sigma(y))-2h(x)-2h(y)}(t) \\ &\quad \geq T_M\{\mu_{f(x+y)+g(x+\sigma(y))-2h(x)-2k(y)}(t), \mu_{2(k(y)-h(y))}(t), \\ &\mu_{f(x+\sigma(y))-g(x+\sigma(y))}(t)\} \\ &\quad \geq T_M\{\Psi(x, y, t), \Psi(0, y, t), \Psi(y, 0, t), \Psi(\frac{x+\sigma(y)}{2}, \frac{x+\sigma(y)}{2}, t), \\ &\quad \Psi(\frac{x+\sigma(y)}{2}, -\frac{x+\sigma(y)}{2}, t)\} \\ &\quad := M(x, y, t). \end{aligned}$$

Therefore

$$\mu_{f(x+y)+f(x+\sigma(y))-2h(x)-2h(y)}(t) \geq M(x, y, t). \tag{20}$$

By putting $y = 0$ in (20), we get

$$\mu_{f(x)-h(x)}(\frac{t}{2}) = \mu_{2(f(x)-h(x))}(t) \tag{21}$$

$$\geq M(x, 0, t). \tag{22}$$

Hence, (20) and (21) imply

$$\begin{aligned} &\mu_{h(x+y)+h(x+\sigma(y))-2h(x)-2h(y)}(t) \\ &\quad \geq T_M\{\mu_{f(x+y)+f(x+\sigma(y))-2h(x)-2h(y)}(t), \mu_{f(x+y)-h(x+y)}(t), \\ &\mu_{f(x+\sigma(y))-h(x+\sigma(y))}(t)\} \\ &\quad \geq T_M\{M(x, y, t), M(x+y, 0, 2t), M(x+\sigma(y), 0, 2t)\} \\ &\quad := M'(x, y, t). \end{aligned}$$

Therefore

$$\mu_{h(x+y)+h(x+\sigma(y))-2h(x)-2h(y)}(t) \geq M'(x, y, t). \tag{23}$$

Now we define X to be the set of all functions $f : X \to Y$ and introduce a generalized metric on X as follows

$$d_{M'}(g, h) = \inf\{\varepsilon \in 0, \infty) : \mu_{g(x)-h(x)}(\varepsilon t) \geq M'(x, x, t), \forall x \in X, \forall t > 0\}. \tag{24}$$

Then, it is easy to verify that $d_{M'}$ is a complete generalized metric on X (see [?]).We define an operator

$J: X \to X$ by

$$JL(x) = \frac{1}{4}\left[L(2x) + L(x + \sigma(x))\right], \tag{25}$$

for all $x \in X$.

First, we assert that J is strictly contractive on X. Given $g, h \in X$, let $\varepsilon \in 0, \infty)$ be an arbitrary constant with $d_{M'}(g, h) \le \varepsilon$, that is,

$$\mu_{g(x)-h(x)}(\varepsilon t) \ge M'(x, x, t), \tag{26}$$

for all $x \in X$. If we replace y by x in (23), then we obtain

$$\mu_{h(2x)+h(x+\sigma(x))-4h(x)}(\varepsilon t) \ge M'(x, x, t), \tag{27}$$

for every $x \in X$. It follows from (26) and (27) that

$$\begin{aligned}
&\mu_{Jg(x)-Jh(x)}\left(\frac{\alpha\varepsilon t}{4}\right) = \mu_{g(2x)+g(x+\sigma(x))-h(2x)-h(x+\sigma(x))}(\alpha\varepsilon t) \\
&\ge T_M\{\mu_{g(2x)-h(2x)}(\alpha\varepsilon t), \mu_{g(x+\sigma(x))-h(x+\sigma(x))}(\alpha\varepsilon t)\} \\
&\ge T_M\{M'(2x, 2x, \alpha t), M'(x+\sigma(x), x+\sigma(x), \alpha t)\} \\
&\ge T_M\{M'(x, x, t), M'(x, x, t)\} \\
&\ge M'(x, x, t),
\end{aligned} \tag{28}$$

for all $x \in X$, that is, $d_{M'}(Jg, Jh) \le \frac{\alpha\varepsilon}{4}$. We hence conclude that

$$d_{M'}(Jg, Jh) \le \frac{\alpha}{4} d_{M'}(g, h)$$

for any $g, h \in X$. Therefore, J is strictly contractive because α is a constant with $0 < \alpha < 4$.

Next, we assert that $d_{M'}(Jh, h) < \infty$. If we put $y = x$ in (23), we get

$$\mu_{Jh(x)-h(x)}\left(\frac{\varepsilon t}{4}\right) = \mu_{\frac{1}{4}[h(2x)+h(x+\sigma(x))]-h(x)}(\varepsilon t) \ge M'(x, x, t), \tag{29}$$

for all $x \in X$, that is,

$$d_{M'}(Jh, h) \le \frac{\alpha}{4} < \infty. \tag{30}$$

By Theorem 2.7, we deduce the existence of a fixed point of J, that is, the existence of a mapping $Q: X \to Y$ which is a fixed point of J, such that $d_{M'}(J^n h, Q) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n h)(x) = \frac{1}{2^{2n}}\left[h(2^n x) + (2^n - 1)h(2^{n-1}x + 2^{n-1}\sigma(x))\right], \tag{31}$$

for each $n \in N$. Also, $d_{M'}(h, Q) \le \frac{1}{1-L} d_{M'}(h, Jh)$ implies the inequality

$$d_{M'}(h, Q) \le \frac{1}{1 - \frac{\alpha}{4}} = \frac{4}{4-\alpha}. \tag{32}$$

If ε_n is a decreasing sequence converging to $\frac{4}{4-\alpha}$, then

$$\mu_{h(x)-Q(x)}(\varepsilon_n t) \ge M'(x, x, t), \tag{33}$$

which implies that

$$\mu_{h(x)-Q(x)}(t) \geq M'(x,x,\frac{t}{\varepsilon_n}). \tag{34}$$

Then by left continuous of $M'(x,x,t)$, we have

$$\mu_{h(x)-T(x)}(t) \geq M'(x,x,\frac{4-\alpha}{4}t). \tag{35}$$

Therefore

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}} \left[h(2^n x) + \left(2^n - 1\right)h(2^{n-1}x + 2^{n-1}\sigma(x))\right], \tag{36}$$

for all $x \in X$. It follows from (8), (35) and (36) that

$$\begin{aligned}
&\mu_{Q(x+y)+Q(x+\sigma(y))-2Q(x)-2Q(y)}(2^{2n}t) \\
&\geq \mu_{h(2^n x+2^n y)+\left(2^n-1\right)h(2^{n-1}(x+y)+2^{n-1}(\sigma(x)+\sigma(y))) +h(2^n x+2^n \sigma(y))+\left(2^n-1\right)h(2^{n-1}(x+\sigma(y))+2^{n-1}(\sigma(x)+y)) -2h(2^n x)+2\left(2^n-1\right)h(2^{n-1}x+2^{n-1}\sigma(x))-2h(2^n y)+2\left(2^n-1\right)h(2^{n-1}y+2^{n-1}\sigma(y))}(t) \\
&\geq T\{\mu_{h(2^n x+2^n y)+h(2^n x+2^n \sigma(y))-2h(2^n x)-2h(2^n y)}(t), \\
&\mu_{h(2^{n-1}(x+\sigma(x))+2^{n-1}(y+\sigma(y)))+h(2^{n-1}(x+\sigma(x))+2^{n-1}(y+\sigma(y))) -2h(2^{n-1}x+2^{n-1}\sigma(x))-2h(2^{n-1}x+2^{n-1}\sigma(x))}(t)\} \\
&\geq T\{M'(2^n x, 2^n x, t), M'(2^n(x+\sigma(x)), 2^n(y+\sigma(y)), t)\} \\
&\geq T\{M'(x,x,\frac{t}{\alpha^n}), M'(x,x,\frac{t}{\alpha^n})\} \\
&= M'(x,x,\frac{t}{\alpha^n}).
\end{aligned}$$

Therefore

$$\mu_{Q(x+y)+Q(x+\sigma(y))-2Q(x)-2Q(y)}(t) \geq M'(x,x,\frac{2^{2n}t}{\alpha^n}) \to 1, \tag{37}$$

for all $x, y \in X$, which implies that Q is a solution of (6).

Assume that $Q_0 : X \to Y$ is an another solution of (8) satisfying (9). We know that Q_0 is a fixed point of J . In view of (9) and the definition of $d_{M'}$, we can conclude that (36) is true with Q_0 in place of Q. Due to Theorem 2.7 (b), we get $Q = Q_0$. This proves the uniqueness of Q.

By (21), (35), we obtain

$$\begin{aligned}
\mu_{f(x)-Q(x)}(t) &\geq T\{\mu_{f(x)-h(x)}(t), \mu_{h(x)-Q(x)}(t)\} \\
&\geq T\{M(x,0,2t), M'(x,x,\frac{4-\alpha}{4}t)\}.
\end{aligned} \tag{38}$$

Also by (18), (35), we obtain

$$\begin{aligned}
\mu_{k(x)-Q(x)}(t) &\geq T\{\mu_{k(x)-h(x)}(t), \mu_{h(x)-Q(x)}(t)\} \\
&\geq T\{\Psi(0, x+\sigma(x), 2t), \Psi(x+\sigma(x), 0, 2t), M'(x,x,\frac{4-\alpha}{4}t)\},
\end{aligned} \tag{39}$$

and by (19), (35), we obtain

$$\mu_{g(x)-Q(x)}(t) \geq T\{\mu_{f(x)-g(x)}(t) + \mu_{f(x)-Q(x)}(t)\}$$
$$\geq T\{M(x,0,2t), M'(x,x,\frac{4-\alpha}{4}t), \Psi(x,x,2t), \Psi(x,-x,2t)\}. \tag{40}$$

This completes the proof.

In the following, we will investigate some special cases of Theorem 3.2.

Corollary 3.3 *Let K be a non-Archimedean field, X be a vector space over K, and (Y,μ,T_M) be a non-Archimedean random Banach space over K. Let $f,g,h,k : X \to Y$ be an even Ψ-approximately Pexider quadratic functional equation with involution. If for some $\alpha \in R$, $0 < \alpha < 8$, such that*

$$\Psi(x,y,t) \geq \Psi(2x,2y,\frac{\alpha}{8}t),$$
$$\Psi(x,y,t) \geq \Psi(x+\sigma(x), y+\sigma(y), \alpha t). \tag{41}$$

in which $x \in X$ and $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_M\{M(x,0,2t), M'(x,x,\frac{8-8\alpha}{\alpha}t)\},$$
$$\mu_{g(x)-Q(x)}(t) \geq T_M\{M(x,0,2t), M'(x,x,\frac{8-8\alpha}{\alpha}t), \Psi(x,x,2t), \Psi(x,-x,2t)\},$$
$$\mu_{h(x)-Q(x)}(t) \geq M'(x,x,\frac{8-8\alpha}{\alpha}t),$$
$$\mu_{k(x)-Q(x)}(t) \geq T_M\{\Psi(0,x+\sigma(x),2t), \Psi(x+\sigma(x),0,2t), M'(x,x,\frac{8-8\alpha}{\alpha}t)\}, \tag{42}$$

for all $x \in X$, and $t > 0$ in which

$$M(x,y,t) = T_M\{\Psi(x,y,t), \Psi(0,y,t), \Psi(y,0,t), \Psi(\frac{x+\sigma(y)}{2}, \frac{x+\sigma(y)}{2}, t), \Psi(\frac{x+\sigma(y)}{2}, -\frac{x+\sigma(y)}{2}, t)\},$$

$$M'(x,y,t) = T_M\{M(x,y,t), M(x+y,0,2t), M(x+\sigma(y),0,2t)\}.$$

Moreover,

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}[h(2^n x) + (2^n - 1)h(2^{n-1}x + 2^{n-1}\sigma(x))].$$

Proof. It is enough to define an operator $J : X \to X$ by

$$JL(x) = 4\left[L(\frac{x}{2}) - \frac{1}{2}L(\frac{x}{4} + \frac{\sigma(x)}{4})\right]. \tag{43}$$

The result will be obtained from the similar argument as in the proof of Theorem 3.2.

4 Random Hyers-Ulam stability in odd cases

In this section, we investigate the stability of the Pexider quadratic functional equation with involution

$$f(x+y) + g(x+\sigma(y)) = 2h(x) + 2k(y) \tag{7}$$

where f, g, h, k odd mappings from X to Y, and $f(0) = g(0) = h(0) = k(0) = 0$, by using the fixed point approach.

Theorem 4.1 *Let K be a non-Archimedean field, X be a vector space over K, and (Y,μ,T_M) be a non-Archimedean random Banach space over K. Let $f,g,h,k:X\to Y$ be odd Ψ-approximately Pexider quadratic functional equation with involution. If for some $\alpha\in R$, $0<\alpha<2$, such that*

$$\Psi(2x,2y,t)\geq\Psi(x,y,\frac{t}{\alpha}),$$
$$\Psi(x+\sigma(x),y+\sigma(y),t)\geq\Psi(x,y,\frac{t}{\alpha}). \tag{44}$$

in which $x,y\in X$ and $t>0$, then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{f(x)-Q(x)}(t)\geq T_M\{\Psi(2d,0,2t),\Psi(0,2d,2t),M'(x,x,\frac{2-\alpha}{8}t)\},$$
$$\mu_{g(x)-Q(x)}(t)\geq T_M\{\Psi(0,y,t),M'(x,x,\frac{2-\alpha}{8}t)\},\Psi(2d,0,2t),\Psi(0,2d,2t)\},$$
$$\mu_{h(x)-Q(x)}(t)\geq M'(x,x,\frac{2-\alpha}{2}t),$$
$$\mu_{k(x)-Q(x)}(t)\geq M'(x,x,\frac{2-\alpha}{2}t), \tag{45}$$

for all $x\in X$ and $t>0$ in which

$$M(x,y,t)=T_M\{\Psi(x,y,t),\Psi(x+y,0,t),\Psi(0,x+y,t),\Psi(2x,0,t),\Psi(0,2x,t),$$
$$\Psi(x,-x,t),\Psi(2y,0,t),\Psi(0,2y,t),\Psi(y,-y,t),\}, \tag{46}$$
$$M'(x,y,t)=T_M\{M(x,y,t),M(x,\sigma(y),t)\}.$$

Moreover,

$$Q(x)=\lim_{n\to\infty}\frac{1}{2^{2n}}[h(2^n x)+(2^n-1)h(2^{n-1}x+2^{n-1}\sigma(x))].$$

Proof. As Theorem 3.2, if we put $y=0$, $x=0$ (and replace y by x), $y=x$, and $y=-x$ in (8) separately, then we obtain

$$\mu_{f(x)+g(x)-2h(x)}(t)\geq\Psi(x,0,t), \tag{47}$$
$$\mu_{f(y)+g(\sigma(y))-2k(y)}(t)\geq\Psi(0,y,t), \tag{48}$$
$$\mu_{f(2x)+g(x+\sigma(x))-2h(x)-2k(x)}(t)\geq\Psi(x,x,t), \tag{49}$$
$$\mu_{g(x-\sigma(x))-2h(x)+2k(x)}(t)\geq\Psi(x,-x,t), \tag{50}$$

for all $x\in X$, respectively.

Since $\sigma:X\to X$ is an involution, we replace y by $x-\sigma(x)$ in (48), then

$$\mu_{f(x-\sigma(x))-g(x-\sigma(x))-2k(x-\sigma(x))}(t)\geq\Psi(0,x-\sigma(x),t). \tag{51}$$

Also, y by $x+\sigma(x)$ in (48), then

$$\mu_{f(x+\sigma(x))+g(x+\sigma(x))-2k(x+\sigma(x))}(t)\geq\Psi(0,x+\sigma(x),t). \tag{52}$$

Also, we replace x by $x-\sigma(x)$ in (49), then

$$\mu_{f(2(x-\sigma(x)))-2h(x-\sigma(x))-2k(x-\sigma(x))}(t)\geq\Psi(x-\sigma(x),x-\sigma(x),t). \tag{53}$$

We replace x by $x-\sigma(x)$ in (50), then

$$\mu_{g(2(x-\sigma(x)))-2h(x-\sigma(x))+2k(x-\sigma(x))}(t)\geq\Psi(x-\sigma(x),-x-\sigma(x),t). \tag{54}$$

Due to (47) and (51), we have

$$\mu_{2f(x-\sigma(x))-2h(x-\sigma(x))-2k(x-\sigma(x))}(t) \geq T_M\{\Psi(x-\sigma(x),0,t),\Psi(0,x-\sigma(x),t)\}, \tag{55}$$

and

$$\mu_{2g(x-\sigma(x))-2h(x-\sigma(x))+2k(x-\sigma(x))}(t) \geq T_M\{\Psi(x-\sigma(x),0,t),\Psi(0,x-\sigma(x),t)\}. \tag{56}$$

Combining (53) with (55) yields

$$\mu_{h(2(x-\sigma(x)))+k(2(x-\sigma(x)))-2h((x-\sigma(x)))-2k((x-\sigma(x)))}(t) \geq T_M\{\Psi(2(x-\sigma(x)),0,\frac{t}{2}),$$
$$\Psi(0,2(x-\sigma(x)),\frac{t}{2}),\Psi(x-\sigma(x),x-\sigma(x),t)\}. \tag{57}$$

Due to (54) and (56), we have

$$\mu_{h(2(x-\sigma(x)))-k(2(x-\sigma(x)))-2h((x-\sigma(x)))+2k((x-\sigma(x)))}(t) \geq T_M\{\Psi(2(x-\sigma(x)),0,\frac{t}{2}),$$
$$\Psi(0,2(x-\sigma(x)),\frac{t}{2}),\Psi((x-\sigma(x)),\sigma(x)-x,t))\}. \tag{58}$$

Now it follows from (57) and (58) that

$$\mu_{h(2(x-\sigma(x)))-2h((x-\sigma(x)))}(t) \geq T_M\{\Psi(2(x-\sigma(x)),0,\frac{t}{4}),\Psi(0,2(x-\sigma(x)),\frac{t}{4}),$$
$$\Psi((x-\sigma(x)),(x-\sigma(x)),\frac{t}{2}),\Psi((x-\sigma(x)),-(x-\sigma(x)),\frac{t}{2})\}, \tag{59}$$

and analogously

$$\mu_{k(2(x-\sigma(x)))-2k((x-\sigma(x)))}(t) \geq T_M\{\Psi(2(x-\sigma(x)),0,\frac{t}{4}),\Psi(0,2(x-\sigma(x)),\frac{t}{4}),$$
$$\Psi((x-\sigma(x)),(x-\sigma(x)),\frac{t}{2}),\Psi((x-\sigma(x)),-(x-\sigma(x)),\frac{t}{2})\}. \tag{60}$$

In view of (8), (55), (56), (59) and (60), we have

$$\begin{aligned}
&\mu_{h(x+y)+k(x+y)+h(x+\sigma(y))-k(x+\sigma(y))-h(2x)-k(2y)}(t)\\
&\geq T_M\{\mu_{f(x+y)+g(x+\sigma(y))-2h(x)-2k(y)}(t),\mu_{h(x+y)+k(x+y)-f(x+y)}(t),\\
&\mu_{h(x+\sigma(y))-k(x+\sigma(y))-g(x+\sigma(y))}(t),\mu_{h(2x)-2h(x)}(t),\mu_{k(2y)-2k(y)}(t)\}\\
&\geq T_M\{\Psi(x,y,t),\Psi(x+y,0,\frac{t}{2}),\Psi(0,x+y,\frac{t}{2}),\Psi(x+\sigma(y),0,\frac{t}{2}),\Psi(0,x+\sigma(y),\frac{t}{2}),\\
&\Psi(2x,0,\frac{t}{4}),\Psi(0,2x,\frac{t}{4}),\Psi(x,x,\frac{t}{2}),\Psi(x,-x,\frac{t}{2}),\Psi(2y,0,\frac{t}{4}),\\
&\Psi(0,2y,\frac{t}{4}),\Psi(y,y,\frac{t}{2}),\Psi(y,-y,\frac{t}{2})\}=M(x,y,t),
\end{aligned} \tag{61}$$

for all $x,y\in X$. If we replace y in (61) by $\sigma(y)$ and then using the fact that h,k is an odd functions, we get

$$\mu_{h(x+\sigma(y))+k(x+\sigma(y))+h(x+y)-k(x+y)-h(2x)-k(2\sigma(y))}(t) \geq M(x,\sigma(y),t).$$

From (61) and (62), we get

$$\mu_{2h(x+y)+2h(x+\sigma(y))-h(2x)-h(2\sigma(y))}(t) \geq T_M\{M(x,y,t),M(x,\sigma(y),t)\}. \tag{62}$$

If we replace y in (61) by $\sigma(y)$ and then using combining (47) with (52), we get

$$\mu_{2h(x+y)+2h(x+\sigma(y))-h(2x)-h(2y)}(t) \geq T_M\{M(x,y,t),M(x,\sigma(y),t)\}. \tag{63}$$

By (59), (60) and (54), we get

$$\mu_{2h(x+y)+2h(x+\sigma(y))-2h(x)-2h(y)}(t) \geq T_M\{M(x,y,t), M(x,\sigma(y),t)\}$$
$$= M'(x,y,t). \tag{64}$$

Therefore

$$\mu_{h(x+y)+h(x+\sigma(y))-h(x)-h(y)}(t) \geq M'(x,y,t). \tag{65}$$

According to Theorem 3.2, we define X to be the set of all functions $f: X \to Y$ and introduce a generalized metric on X as follows.

$$d_{M'}(g,h) = \inf\{\varepsilon \in 0,\infty) : \mu_{g(x)-h(x)}(\varepsilon t) \geq M'(x,x,t), \forall x \in X, \forall t > 0\}. \tag{66}$$

Also, We define an operator $J: X \to X$ by

$$JL(x) = \frac{1}{2}[L(2x) + L(x+\sigma(x))], \tag{67}$$

for all $x \in X$.

Now, similar to that of the proof of Theorem 3.2, there exists a function $Q: X \to Y$ which is a fixed point of J such that $d_{M'}(J^n h, Q) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n h)(x) = \frac{1}{2^n}\left[h(2^n x) + (2^n - 1)h(2^{n-1}x + 2^{n-1}\sigma(x))\right], \tag{68}$$

for each $n \in N$. Since $d_{M'}(J^n h, Q) \to 0$ as $n \to \infty$, there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to \frac{2}{2-\alpha}$ as $n \to \infty$ and $d_{M'}(J^n h, Q) \leq \varepsilon_n$ for every $n \in N$. Hence, it follows from the definition of $d_{M'}$ that

$$\mu_{J^n h(x)-Q(x)}(\varepsilon_n t) \geq M'(x,x,\frac{2-\alpha}{2}t), \tag{69}$$

for all $x \in X$. Therefore

$$Q(x) = \lim_{n\to\infty}\frac{1}{2^n}\left[h(2^n x) + (2^n - 1)h(2^{n-1}x + 2^{n-1}\sigma(x))\right] \tag{70}$$

for all $x \in X$. Similarly to Theorem 3.2, Q is a solution of (6).

In the following, we will investigate special cases of Theorem 4.1.

Corollary 4.2 *Let K be a non-Archimedean field, X be a vector space over K, and (Y, μ, T_M) be a non-Archimedean random Banach space over K. Let $f, g, h, k: X \to Y$ be odd Ψ-approximately Pexider quadratic functional equation with involution. If for some $\alpha \in R$, $0 < \alpha < 4$, such that*

$$\Psi(x,y,t) \geq \Psi(2x,2y,\frac{\alpha}{4}t),$$
$$\Psi(x+\sigma(x), y+\sigma(y), t) \geq \Psi(x,y,2t). \tag{71}$$

in which $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $Q: X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_M\{\Psi(2d,0,2t), \Psi(0,2d,2t), M'(x,x,\frac{2-2\alpha}{\alpha}t)\},$$

$$\mu_{g(x)-Q(x)}(t) \geq T_M\{\Psi(2d,0,2t), M'(x,x,\frac{2-2\alpha}{\alpha}t), \Psi(2d,0,2t), \Psi(0,y,t)\},$$

$$\mu_{h(x)-Q(x)}(t) \geq M'(x,x,\frac{2-2\alpha}{\alpha}t),$$

$$\mu_{k(x)-Q(x)}(t) \geq M'(x,x,\frac{2-2\alpha}{\alpha}t), \tag{72}$$

for all $x \in X$ and $t > 0$ in which

$$\begin{aligned} M(x,y,t) &= T_M\{\Psi(x,y,t),\Psi(x+y,0,t),\Psi(0,x+y,t),\Psi(2x,0,t),\Psi(0,2x,t), \\ &\Psi(x,-x,t),\Psi(2y,0,t),\Psi(0,2y,t),\Psi(y,-y,t)\}, \\ M'(x,y,t) &= T_M\{M(x,y,t),M(x,\sigma(y),t)\}. \end{aligned} \tag{73}$$

Moreover,

$$Q(x) = \lim_{n\to\infty} 2^n[h(\frac{x}{2^n}) + \left(\frac{1}{2^n}-1\right)h(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})].$$

Proof. It is enough to define an operator $J : X \to X$ by

$$JL(x) = 2\left[L(\frac{x}{2}) - \frac{1}{2}L(\frac{x}{4}+\frac{\sigma(x)}{4})\right]. \tag{74}$$

The result will be obtained from the similar argument as in the proof of Theorem 4.1.

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