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A Random Variable Distributed between Two Random Variables

Running title: A Random Variable Distributed between Two Random Variables

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ABSTRACT

Let X and Y be independent random variables and let Z be a random variable (which is uniform or not uniformly distributed) over [X,Y]. We study the distribution of the random variable Z and show that the arcsine distribution and Cauchy distribution can be characterized in a particular way by means of this construction.

KEYWORDS: Arcsin, Cauchy, Stieltjes Transform, Schwartz theory.

1. INTRODUCTION

In a fundamental paper, Van Assche (1987) considered a random variable Z^* uniformly distributed between two independent random variables X and Y; in the sense that the conditional distribution of Z^* given X=x and Y=y is uniform over [min {x,y}, max {x,y}]. Mathematically this means that

$$p(Z^* \leq z | X = x, Y = y) = \begin{cases} 1, & z \geq max(y,x) \\ \dfrac{z-x}{y-x}, & x < z < y \\ \dfrac{z-y}{x-y}, & y < z < x \\ 0, & z \leq min(y,x) \end{cases}$$

By applying certain properties of the distributional derivatives, Van Assche (1987) derived the following interesting results. Assuming X and Y are identically distributed, then

Result (i):Under the assumption that X, Y are independent and continuous with distributions F_X, $F_Y$and $F_Z^*$is distribution of Z^*,relation between stieltjes transforms distributions of F_Z^*, $F_X$and$F_Y$is given that

$$-S'(F_Z^*, z) = S(F_X, z)S(F_Y, z)$$

Result (ii): for X and Y on $[-1, 1]$, $Z^*$is uniform on $[-1, 1]$ if and only if X and Yhaveanarcsin distribution;

Result (iii): Z^*possesses the same distribution as X and Y if and only if X and Y are degenerated or have a Cauchy distribution.

Kotzand Johnson have cleverly done a further study of Van Assche's work. But they did not succeed to get the intended result(2). It also should be mentioned that the powerful method of Van Assche for calculation of distribution compared to Johnson and Kotz [1] method is more flexible. In recent years, several papers are presented on weighted average (see Johnson and kotz [1], Soltani-homei [5]). By using random division Soltani and Homei extended Van Assch –Johnson and kotz's work and extended weighted average. But it seemed their result hold only for the case of n=2 and even for getting the cauchy character one should solve an equation of order n that is very difficult. The special case of n=2, is equivalent to choosing a sample from uniform distribution and getting (W,1-W). It seems that uniform distribution in Van Assche's paper is important. Main aim in paper is to show the result given Van Assche is not just for uniform distribution a lone. In Section 2, we introduced the distribution similar construction of random variable Z* uniformly distributed between two independent random variables X and Y.In Section 3, we establish the 2-th derivative of the Stieltjes transform of the distribution of Z is expressed in terms of the product of the Stieltjes transforms of the distributions one and derivative of other, Theorem 3.1.In Section 4, As, we observe that (Van Assche) two characterization for the Cauchy and arcsin distribution are true.

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2 A Random Variable Distributed Between Two Independent Random Variables X And Y

Suppose that X and Y are two independent random variables (which are not necessarily identically distributed). We suppose that Z is a random variable between X and Y with conditional distribution, given X = x, Y = y ;

$$F_{z|x,y}(z) = \begin{cases} 1, & z \geq \max(y,x) \\ \left(\frac{z-x}{y-x}\right)^2, & x < z < y \\ -2\left(\frac{z-y}{y-x}\right) - \left(\frac{z-y}{y-x}\right)^2, & y < z < x \\ 0, & z \leq \min(y,x), \end{cases}$$

The conditional distribution of Z, given X = x and Y = y, can be written in a more compact form using the Heaviside function:

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Namely

$$P(Z \leq z|X = x, Y = y) = \left(\frac{z-x}{y-x}\right)^2 U(z-x) + 2\left(\frac{z-y}{x-y}\right) U(z-y) - \left(\frac{z-y}{x-y}\right)^2 U(z-y). \quad (1)$$

The distribution function of Z can then be written down by integrating the X and Y out, giving

$$P(Z \leq z) = \int_{\mathbb{R}^2} \left(\frac{z-x}{y-x}\right)^2 U(z-x) + 2\left(\frac{z-y}{x-y}\right) U(z-y) - \left(\frac{z-y}{x-y}\right)^2 U(z-y) dF_X\, dF_Y \quad (2)$$

Where, the distribution function of Z, X and Y, respectively are F_Z , $F_X \, and \, F_Y$.

3 Relation Between Stieltjes Transforms

Let us denote the Stieltjes transform of a distribution K by

$$S(K,z) = \int_{\mathbb{R}} \frac{1}{z-x} K(dx), \quad (3)$$

for every z in the set of complex numbers ℂ which does not belong to the support of K, z ∈ℂ ∩ (suppK)^c. For more on the Stieltjes transform see [7]. Let us also barrow the following tools from the Schwartz theory on distributional derivatives. Indeed every distribution Λ is viewed as a Schwartz distribution on a certain infinitely differentiable functions φ through $\int_{-\infty}^{\infty} \varphi(x)\Lambda(dx)$. It is well know that Λ admits a derivative of order 2, denoted by $\Lambda^{[2]}$, possessing

$$\int_{-\infty}^{\infty} \varphi(x)\Lambda^{[2]}(dx) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d^2}{dx^2}\varphi(x)\Lambda(dx). \quad (4)$$

A distribution $\Lambda^{[2]}$ that possesses (4) is called the 2-th distributional derivative of Λ.

In this section we present relation between stieltjes transforms distributions of F_Z, F_X and F_Y.

Theorem 3.1. Under the assumption that X,Y are independent and continuous, relation between stieltjes transforms distributions is given that

$$-\frac{1}{2} S''(F_Z,z) = S(F_X,z)S'(F_Y,z) \quad (5)$$

Where $S''(F_Z,z)$ and $S'(F_Y,z)$ are second and first derivative of S(., z) on z.

Proof. The conditional distribution P(Z ≤ z | X = x, Y = y) given by (1) leads us to the following linear functional on complex-valued functions , defined on the set of real numbers R;

$$P(Z \leq z|X = x, Y = y) = \left(\frac{z-x}{y-x}\right)^2 U(z-x) + 2\left(\frac{z-y}{x-y}\right) U(z-y) - \left(\frac{z-y}{x-y}\right)^2 U(z-y). \quad (6)$$

It easily follows that

$$P(Z \leq z|X = x, Y = y) = \frac{f_z(x)}{(y-x)^2} + \frac{\frac{df_z(y)}{dz}}{x-y} - \frac{f_z(y)}{(x-y)^2} \quad (7)$$

Where $f_z(x) = (z-x)^2 U(z-x)$. Also we note that

$$U(z-x) = \frac{1}{2}\frac{d^2}{dx^2} f_z(x)$$

Thus

$$P(Z \le z) = \int_{\mathbb{R}} U(z-x)dF_z(x) = \int_{\mathbb{R}^2}\left(\frac{f_z(x)}{(y-x)^2} + \frac{\frac{df_z(y)}{dz}}{x-y} - \frac{f_z(y)}{(x-y)^2}\right)dF_X\, dF_Y$$

can be viewed as:

$$\frac{1}{2}\int_{\mathbb{R}} \frac{d^2}{dx^2} f_Z(x)dF_Z(x) = \int_{\mathbb{R}^2}\left(\frac{f_z(x)}{(y-x)^2} + \frac{\frac{df_z(y)}{dz}}{x-y} - \frac{f_z(y)}{(x-y)^2}\right)dF_X\, dF_Y \qquad (8)$$

Now (8) together with (4) lead us to

$$\int_{\mathbb{R}^2} f_z(x)df_z^{[2]}(\mathrm{x}) = \int_{\mathbb{R}^2} \frac{f_z(x)}{(y-x)^2} + \frac{\frac{df_z(y)}{dz}}{x-y} - \frac{f_z(y)}{(x-y)^2}\, dF_X\, dF_Y (9)$$

where$F_Z^{[2]}$is the (2)-th distributional derivative of the distribution F_Z.

Therefore by using the linear property (6) and a standard argument in the integration theory, we obtain that

$$\int_{\mathbb{R}} f\ \ (x)dF_z^{[2]}(\mathrm{x}) = \int_{\mathbb{R}^2} \frac{f\ \ (x)}{(y-x)^2} + \frac{\frac{df\ \ (y)}{dz}}{x-y} - \frac{f\ \ (y)}{(x-y)^2}\, dF_X\, dF_Y (10)$$

for a suitable f.

We follows from (10) that, *for* $f(x) = \frac{1}{\mathrm{z-x}}$

$$\int_{\mathbb{R}} f\ \ (x)dF_z^{[2]}(x) = \int_{\mathbb{R}^2} \frac{\frac{1}{\mathrm{z-x}}}{(y-x)^2} + \frac{-\frac{1}{(\mathrm{z-y})^2}}{x-y} - \frac{\frac{1}{\mathrm{z-y}}}{(x-y)^2}\, dF_X\, dF_Y$$

$$= \int_{\mathbb{R}^2} \frac{\frac{1}{(y-x)^2}}{(\mathrm{z-x})} + \frac{-\frac{1}{x-y}}{(\mathrm{z-y})^2} - \frac{\frac{1}{(x-y)^2}}{\mathrm{z-y}}\, dF_X\, dF_Y$$

$$= \int_{\mathbb{R}^2} \frac{1}{(\mathrm{z-x})}\frac{1}{(\mathrm{z-y})^2}\, dF_X\, dF_Y$$

Thus

$$-S\left(F_Z^{[2]}, z\right) = S(F_X, z)S'(F_Y, z), z\epsilon\mathbb{C} \cap (supp\ F_X \cap suppF_Y)^c \qquad (11)$$

Therefore

$$\frac{d^2}{dz^2}S(F_z, z) = \int_{\mathbb{R}} \frac{2}{(z-x)^3}F_z(dx)$$
$$= \int_{\mathbb{R}} \frac{d^2}{dx^2}\frac{1}{z-x}F_z(dx)$$
$$= 2\int_{\mathbb{R}} \frac{1}{z-x}F_z^{[2]}(dx)$$
$$= 2S\left(F_z^{[2]}, z\right)$$
$$= -2S(F_X, z)S'(F_X, z).$$

giving the result. The proof of the theorem is complete.

4 Characterization

Now we are to review Van Assche's findings.

Theorem 4.1. Let X and Y be i.i.d random variables on [-1,1], then Z is uniformly distributed on [-1,1] if and only if X and Y have an arcsin distribution on [-1,1], i.e,

$$P(Z \le t) = P(Y \le t) = \frac{1}{\pi}\int_{-1}^{t}\frac{1}{\sqrt{1-x^2}}dx.$$

Proof . The random variable Z has a uniform distribution on [-1,1] and F is the distribution function of X and Y, then it follows from the theorem 3.1

$$S(F_X, z)S'(F_Y, z) = -\frac{1}{2}S''(F_Z, z)$$

$$= -\frac{z}{(z^2-1)^2}$$

The solution$S(F_X, z)$is

$$S(F_X, z) = \frac{1}{\sqrt{z^2 - 1}}$$

Which is the stieltjes transform of the arcsindistribution.

Theorem 4.2. Let X and Y be i.i.d. random variables, then Z has the same distribution as X and Y if and only if X and Y are almost surely constant or have a Cauchy distribution.

Proof. The random variable Z has the same distribution as X and Y, so that we have to find the idempotent probability measures for F, here F is the probability distribution function of X and Y. Then it follows from the theorem 3.1,

$$-\frac{1}{2}S"(F, z) = S(F, z)S'(F, z)$$

By using a Reduction Order in ODE method the solution is obtain as

$$S(F, z) = \frac{1}{z + c}$$

Where c is a constant.In order that this is the stieltjes transform of a probability distribution, weneed to have that Im$S(F, z) < 0$ whenever Im$(z) > 0$, $b \neq 0$, Therefore

$$S(F, z) = \frac{1}{z - a + ib} \quad Im\,(z) > 0, b \neq 0$$

Where a is real and $b \geq 0$.the case $b = 0$ corresponds to F(x) = U(x − a) whence X and Y are almost surely constant. When b> 0 we have

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{b}{b^2 + (t - a)^2} dt$$

Which is the Cauchy distribution with center parameter a and spread parameter$b$.

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