

Determining the Best Game Cross Efficiency with VRS for Fuzzy Data

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ABSTRACT

Cross efficiency links Decision Making Units (DMUs) performance together. The main idea of cross efficiency is to use Data Envelopment Analysis (DEA) in a peer evaluation. Several methods have been proposed to obtain cross efficiency. These algorithms need a lot of computation efforts. The current paper provides a model to compute the best game cross efficiency for DMUs with Variable Returns to Scale (VRS) for fuzzy data by solving only one model. We deal with linear programming problems with fuzzy parameters from the viewpoint of expert's imprecise or fuzzy understanding of the nature of parameters in a problem- formulation process and consider fuzzy linear programming games arising from the linear programming problems. The proposed method can be used to determine the cross efficiency of DMUs with Constant Returns to Scale (CRS) for fuzzy data. Finally, the proposed model is applied to a numerical example.

KEYWORDS: Data Envelopment Analysis (DEA), Cross Efficiency, Game Cross Efficiency, Fuzzy Data, Fuzzy Game.

1. INTRODUCTION

Data Envelopment Analysis (DEA), introduced by Charnes et al. in 1978 [2], is a mathematical programming based approach that evaluates the efficiency of an organization or, in general, a Decision Making Unit (DMU) relative to a set of comparable organizations. DEA considers multiple inputs and outputs simultaneously, requiring neither a priori weights nor a functional form for input/output relationships. The cross efficiency score for a DMU is obtained by computing that DMUs set of n scores, and then averaging those scores. The main idea of cross efficiency is to use DEA in a peer evaluation [7, 3]. Cross efficiency is use to differentiate between good and poor performers. As mentioned by Doyle and Green [3], cross efficiency is a process with the concept of peer appraisal, as opposed to self appraisal implied by simple efficiency. As demonstrated in [3], because of the non-uniqueness of the DEA optimal weights, the methods of aggressive and benevolent were proposed. Depending on which of the alternative optimal solution to the DEA linear programs is used, it may be possible to improve a DMUs performance rating, but generally only by worsening the rating of others. Aggressive (benevolent) model not only maximize the efficiency of a particular DMU under evaluation, but also minimize (maximize) the average efficiency of other DMUs.

As pointed out by Liang et al. [5], the game cross efficiency model was represented. Liang et al. [5] presented a procedure under CRS assumption for determining the best average game cross efficiency for DMUs. This procedure converges after some iteration. When the number of DMUs or the number of iterations of this procedure increased, computation efforts increase as well. This paper proposes a method to compute the best game cross efficiency under Variable Returns to Scale (VRS), which reduces the number of computations efforts for fuzzy data. In most real-world situations, the possible values of parameters of mathematical models are often only imprecisely or ambiguously known to the experts. It would be certainly more appropriate to interpret the experts understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy sets of the real line known as fuzzy numbers. The proposed method can be used to DMUs with Constant Retunes to Scale (CRS) with fuzzy data. Then we demonstrate proposed model with a numerical example. The rest of current paper is organized as follows. Section 2 provides a background of DEA and DEA cross efficiency for fuzzy data. Section 3 proposes the new game cross efficiency model under VRS assumption for fuzzy data. Section 4 discusses an example that applies the proposed method. Finally, section 5 presents concluding remarks.

2. DEA CROSS EFFICIENCY IN FUZZY ENVIROMENT

Let us suppose a production technology transforming a series of input vectors $\vec{x}_{ij} = (L_{ij}, x_{ij}, U_{ij})$ (i = 1, ..., m) into

the following output vectors $\tilde{y}_{ij} = (L'_{ij}, y_{ij}, U'_{ij})$ (r = 1, ..., s) where the subscript j (j = 1, ..., n) refers to a set of observed production processes – e.g. country, regional or local economies. The efficiency rating for any given DMU*d* is computed using the BCC model that was rendered by Banker et al [2]:

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$$Max \quad \theta_{d} = \sum_{r=1}^{s} \mu_{r} \tilde{y}_{u} - u_{o}$$

$$st \quad \sum_{i=1}^{m} w_{i} \tilde{x}_{ij} - \sum_{r=1}^{s} \mu_{r} \tilde{y}_{u} + u_{o} \ge 0 \ j = 1, ..., n$$

$$\sum_{i=1}^{m} w_{i} \tilde{x}_{id} = 1, \ w_{i} \ge 0 \ i = 1, ..., m$$

$$\mu_{n} \ge 0 \ r = 1, ..., s, \ u_{o} \ free$$
(1)

Where \tilde{y}_{rd} and \tilde{x}_{ij} are triangular fuzzy number, in this paper, we employ a parametric approach to solving the linear programming problem with fuzzy parameters in order to construct the values of coalitions [6]. First we introduce the α -level and α' of the fuzzy numbers \tilde{y}_{rd} and \tilde{x}_{ij} defined as the set $(\tilde{y}_{rd})_{\alpha}$ and $(\tilde{x}_{ij})_{\alpha}$ in which the degree of their membership functions exceeds the level α and α' :

$$(\tilde{y}_{rd})_{\alpha'} = \{ (y_{rd}) \ \mu_{\tilde{y}_{rd}}(y_{rd}) \ge \alpha', \ (\tilde{x}_{ij})_{\alpha} = \{ (x_{ij}) \ \mu_{\tilde{x}_{ij}}(x_{ij}) \ge \alpha, \\ d = 1, \dots, n, r = 1, \dots, s \} \qquad j = 1, \dots, n, i = 1, \dots, m \}$$

$$(2)$$

Now suppose that all players consider that the degree of all the membership functions of the fuzzy number involved in the linear programming problem should be greater than or equal to a certain degree α and α' . Then, for such a degree α and α' , the problem can be interpreted as the following non fuzzy linear programming problem which depends on a coefficient vector $(x_{ij}) \in (\tilde{x}_{ij})_{\alpha}$ and $(y_{rd}) \in (\tilde{y}_{rd})_{\alpha'}$ [6].

For a certain degree α and α' , it seems to be quite natural to have understood the linear programming problem with fuzzy parameters as the following non fuzzy problem:

$$Max \quad \theta_{d} = \sum_{r=1}^{3} \mu_{r} y_{n} - u_{\circ}$$

$$st \quad \sum_{i=1}^{m} w_{i} x_{ij} - \sum_{r=1}^{s} \mu_{r} y_{n} + u_{\circ} \ge 0 \ j = 1, ..., n$$

$$\sum_{i=1}^{m} w_{i} x_{n} = 1, \quad w_{i} \ge 0 \quad i = 1, ..., m$$

$$\mu_{r} \ge 0 \quad r = 1, ..., s, \quad u_{\circ} \ free$$

$$(x_{ij}) \in (\tilde{x}_{ij})_{\alpha} , \quad (y_{rd}) \in (\tilde{y}_{rd})_{\alpha'}$$
(3)

It should be noted that the coefficient vector (x_{ik}) and (y_{nl}) are treated as decision variables rather than constants. Therefore, the problem (3) is not a linear programming problem. However, from the properties of the α -level and α' -level sets for the vectors of fuzzy number \tilde{x} and \tilde{y} it follows that the feasible regions for \tilde{x} and \tilde{y} can be denoted respectively by the closed interval $[x^{L}, x^{R}]$ and $[y^{L}, y^{R}]$. Thus, we can obtain an optimal solution to the problem (3) by solving the following linear programming problem [4]:

$$Max \quad \theta_{d} = \sum_{r=1}^{s} \mu_{r} y_{n}^{R} - u_{o}$$

$$st \quad \sum_{i=1}^{m} w_{i} x_{ij}^{R} - \sum_{r=1}^{s} \mu_{r} y_{ij}^{R} + u_{o} \ge 0 \ j = 1, ..., n$$

$$\sum_{i=1}^{m} w_{i} x_{id}^{R} = 1, \quad w_{i} \ge 0 \ i = 1, ..., m$$

$$\mu_{r} \ge 0 \ r = 1, ..., s, \quad u_{o} \ free$$

$$(4)$$

Then taking opposite extreme points of the closed interval $[x^{L}, x^{R}]$ and $[y^{L}, y^{R}]$, we can formulate the following problem which yields a value of the objective function smaller than that the problem (5):

$$Max \quad \theta_{d} = \sum_{r=1}^{s} \mu_{r} y_{n}^{L} - u_{o}$$

$$st \sum_{i=1}^{m} w_{i} x_{ij}^{L} - \sum_{r=1}^{s} \mu_{r} y_{j}^{L} + u_{o} \ge 0 \ j = 1, ..., n$$

$$\sum_{i=1}^{m} w_{i} x_{id}^{L} = 1, \ w_{i} \ge 0 \ i = 1, ..., m$$

$$\mu_{r} \ge 0 \ r = 1, ..., s, \ u_{o} \ free$$
(5)

Let $(w_1^{\alpha kL}, ..., w_m^{\alpha kL})$, $(w_1^{\alpha kR}, ..., w_m^{\alpha kR})$ and $(\overline{w_1^{\alpha' kL}}, ..., \overline{w_m^{\alpha' kR}})$, $(\overline{w_1^{\alpha' kR}}, ..., \overline{w_m^{\alpha' kR}})$ denote optimal solution to the problem (5) and (6), respectively for given α and α' [4].

We obtain a set of optimal weights for each DMUd (d = 1,...,n). The *d* cross efficiency for any DMUj (j = 1,...,n), using the optimal set is then calculated as:

$$E_{dj} = \frac{\sum_{i=1}^{m} \mu_{id}^{*} \tilde{y}_{ij} - u_{i}^{*}}{\sum_{i=1}^{m} w_{id}^{*} \tilde{x}_{ij}}, \ d, j = 1, ..., n$$
(6)

The average of all E_{dj} (d = 1,...,n) is a new efficiency measure for DMUj (j = 1,...,n) and is used as the cross efficiency score [5]:

$$\bar{E}_{j} = \frac{1}{n} \sum_{d=1}^{n} \tilde{E}_{dj}, \ j = 1, ..., n$$
(7)

As noted in Sexton et al. [5], the optimal weights obtained from model (6) may not be unique. One remedy suggested is to introduce a secondary objective function to resolve ambiguity. Doyle and Green [3] introduced the aggressive and benevolent model. One version of their model seeks to find a multiplier bundle that maximizes the average of the efficiency ratio of the other *n*-1DMUs with the constraint that the ratio for DMU*d* stays at or above its predetermined optimal level. Specially, cross efficiency provides for a measure of efficiency that not only the best multiplier bundle for DMU*d* under evaluation, but also the best bundles for all other DMUs. Liang et al [5] proposed the game cross efficiency model. In their model, rather than using the ideal score for DMU*d*, they strive to use a score which will actually be representative of its final measure of performance. They defined the game cross efficiency in fuzzy environment for DMU*j* relative to DMU*d* as [5]:

$$\alpha_{dj} = \frac{\sum_{i=1}^{3} \mu_{ij}^{d} \, \tilde{y}_{ij} - u_{.}^{d}}{\sum_{i=1}^{m} w_{ij}^{d} \tilde{x}_{ij}}, \ d = 1, ..., n$$
(8)

Where μ_{nd} and w_{ij} are optimal weights in the following model. To compute the *d* fuzzy game cross efficiency proposed the following mathematical programming for each DMU*j* [3].

$$Max \sum_{r=1}^{m} \mu_{j}^{d} \tilde{y}_{j}$$

$$st \sum_{i=1}^{m} w_{ij}^{d} \tilde{x}_{il} - \sum_{r=1}^{s} \mu_{j} \tilde{y}_{rl} \ge 0l = 1,...,n$$

$$\sum_{i=1}^{m} w_{ij}^{d} x_{il} = 1 \qquad \alpha_{d} \times \sum_{i=1}^{m} w_{ij}^{d} \tilde{x}_{id} - \sum_{r=1}^{s} \mu_{j}^{d} \tilde{y}_{rd} \le 0$$

$$w_{ij}^{d} \ge 0 \quad i = 1,...,m, \quad \mu_{a}^{d} \ge 0 \quad r = 1,...,s$$
(9)

Similar, program (3) we have:

$$Max \sum_{r=1}^{m} \mu_{j}^{d} y_{j}^{R}$$

$$st \sum_{i=1}^{m} w_{ij}^{d} x_{u}^{R} - \sum_{r=1}^{s} \mu_{j} \tilde{y}_{d}^{R} \ge 0l = 1, ..., n$$

$$\sum_{i=1}^{m} w_{ij}^{d} x_{u}^{R} = 1 \qquad \alpha_{d} \times \sum_{i=1}^{m} w_{ij}^{d} x_{u}^{R} - \sum_{r=1}^{s} \mu_{j}^{d} \tilde{y}_{nl}^{R} \le 0$$

$$w_{ij}^{d} \ge 0 \quad i = 1, ..., m, \quad \mu_{j}^{d} \ge 0 \quad r = 1, ..., s$$
(10)

$$Max \sum_{r=1}^{m} \mu_{j}^{d} y_{j}^{L}$$

$$st \sum_{i=1}^{m} w_{ij}^{d} x_{ii}^{L} - \sum_{r=1}^{s} \mu_{j} \tilde{y}_{il}^{L} \ge 0l = 1, ..., n$$

$$\sum_{i=1}^{m} w_{ij}^{d} x_{ii}^{L} = 1 \qquad \alpha_{d} \times \sum_{i=1}^{m} w_{ij}^{d} x_{id}^{L} - \sum_{r=1}^{s} \mu_{j}^{d} \tilde{y}_{rd}^{L} \le 0$$

$$w_{ij}^{d} \ge 0 \quad i = 1, ..., m, \quad \mu_{j}^{d} \ge 0 \quad r = 1, ..., s$$
(11)

Where $\alpha_d \leq 1$ is a parameter [3] also rendered an algorithm for deriving average game cross efficiency score. Model (11) maximizes the efficiency of DMU*j* under condition that the ratio efficiency of DMU*d* isn't less than its original average cross efficiency. Then the average fuzzy game cross efficiency for each DMU*j* was defined as:

$$\alpha_{j} = \frac{1}{n} \sum_{d=1}^{n} \sum_{r=1}^{s} \mu_{rj}^{d*}(\alpha_{d}) \tilde{y}_{rj}, \quad (12)$$

Similarly we have,

$$\alpha_{j} = \frac{1}{n} \sum_{d=1}^{n} \sum_{r=1}^{s} \mu_{d}^{d*}(\alpha_{d}) y_{d}^{L}, \quad \alpha_{j} = \frac{1}{n} \sum_{d=1}^{n} \sum_{r=1}^{s} \mu_{d}^{d*}(\alpha_{d}) y_{d}^{R}, \quad (13)$$

3. THE PROPOSED MODEL TO OBTAIN THE BEST CROSS EFFICIENCY FOR FUZZY DATA

This section presents a model to reduce the computation efforts of the proposed method [4]. That model maximizes the efficiency of DMU*j* under the condition that the difference between ratio efficiency of DMU*d* and α_d isn't less than zero, where α_d is average cross efficiency in its first usage. In our model we want to find the optimal weights for each DMU*j* and also minimize the difference between simple efficiency of DMU*j* and α_{jd} (d = 1, ..., n). For this purpose we add a constraint, the difference between simple efficiency and cross efficiency is less than α , to the CCR model and also add the variable α with negative sign to the objective function. We consider the following mathematical fuzzy programming problem for each DMU*j*:

$$\begin{aligned} &Max \ \sum_{r=1}^{s} \mu_{ij}^{d} \, \tilde{y}_{ij} - u_{.}^{d} - \alpha \\ &s \ t \ \sum_{r=1}^{s} \mu_{ij}^{d} \, \tilde{y}_{il} - \sum_{i=1}^{m} w_{ij}^{d} \, \tilde{x}_{il} - u_{.}^{d} \leq 0 \quad l = 1, ..., n \\ &\sum_{i=1}^{m} w_{ij}^{d} \, \tilde{x}_{ij} = 1 \quad , \theta_{d} - \frac{\sum_{r=1}^{s} \mu_{ij}^{d} \, \tilde{y}_{rd} - u_{.}^{d}}{\sum_{i=1}^{m} w_{ij}^{d} \, \tilde{x}_{id}} \leq \alpha \\ &w_{ij}^{d} \geq 0 \ i = 1, ..., m, \\ &\mu_{ij}^{d} \geq 0 \ r = 1, ..., s, u_{.}^{d} \ free \end{aligned}$$
(14)

Similar, program (3) we have:

$$\begin{aligned} Max \ \sum_{r=1}^{s} \mu_{\sigma}^{d} y_{\sigma}^{R} - u_{*}^{d} - \alpha \\ st \ \sum_{r=1}^{s} \mu_{\sigma}^{d} y_{d}^{R} - \sum_{i=1}^{m} w_{ij}^{d} x_{ii}^{R} - u_{*}^{d} \leq 0 \ l = 1, ..., n \\ \\ \sum_{i=1}^{m} w_{ij}^{d} x_{ij}^{R} = 1 \quad , \theta_{d} - \frac{\sum_{r=1}^{s} \mu_{\sigma}^{d} y_{rd}^{R} - u_{*}^{d}}{\sum_{i=1}^{m} w_{ij}^{d} x_{id}^{R}} \leq \alpha \\ \\ w_{ij}^{d} \geq 0 \ i = 1, ..., m, \\ \\ \mu_{ji}^{d} \geq 0 \ r = 1, ..., s, u_{*}^{d} free \end{aligned}$$

$$(15)$$

$$\begin{aligned} Max & \sum_{r=1}^{s} \mu_{d}^{d} y_{d}^{L} - u_{*}^{d} - \alpha \\ s t & \sum_{r=1}^{s} \mu_{d}^{d} y_{d}^{L} - \sum_{i=1}^{m} w_{ij}^{d} x_{il}^{L} - u_{*}^{d} \leq 0l = 1, ..., n \\ & \sum_{i=1}^{m} w_{ij}^{d} x_{ij}^{L} = 1 \quad , \theta_{d} - \frac{\sum_{i=1}^{s} \mu_{d}^{d} y_{id}^{L} - u_{*}^{d}}{\sum_{i=1}^{m} w_{ij}^{d} x_{id}^{L}} \leq \alpha \\ & w_{ij}^{d} \geq 0 \quad i = 1, ..., m, \\ & \mu_{dj}^{d} \geq 0 \quad r = 1, ..., s, u_{*}^{d} free \end{aligned}$$
(16)

Where $\theta_j \square$ is the simple efficiency of DMU*j*, for each DMU*j*, this model is solve *n* times, once for each d = 1, ..., n. Let μ_{ij}^{d*} and w_{ij}^{d*} be optimal solution to model (15, 16), thus the game cross efficiency is defined as:

$$\alpha_{jd} = \frac{\sum_{i=1}^{s} \mu_{ij}^{d*} \tilde{y}_{u} - u_{o}^{d*}}{\sum_{i=1}^{m} w_{ij}^{d*} \tilde{x}_{u}}, \ d = 1, ..., n$$
(17)

Similar, program (3) we have:

$$\alpha_{jd} = \frac{\sum_{i=1}^{s} \mu_{ij}^{d*} y_{id}^{R} - u_{i}^{d*}}{\sum_{i=1}^{m} w_{ij}^{d*} x_{id}^{R}}, \quad d = 1, ..., n \quad (18) \qquad \alpha_{jd} = \frac{\sum_{i=1}^{s} \mu_{ij}^{d*} y_{id}^{L} - u_{i}^{d*}}{\sum_{i=1}^{m} w_{ij}^{d*} x_{id}^{L}}, \quad d = 1, ..., n \quad (19)$$

Therefore, the average game cross efficiency for each DMU*j* is defined as:

$$\alpha_{j} = \frac{1}{n} \sum_{d=1}^{n} \frac{\sum_{j=1}^{s} \mu_{ij}^{d*} \tilde{y}_{n} - \mu_{-}^{d*}}{\sum_{i=1}^{m} w_{ij}^{d*} \tilde{x}_{id}}, \quad j = 1, ..., n$$
(20)

Similar, program (3) we have:

$$\alpha_{j} = \frac{1}{n} \sum_{d=1}^{n} \frac{\sum_{r=1}^{s} \mu_{rj}^{d*} y_{n}^{R} - u_{s}^{d*}}{\sum_{i=1}^{m} w_{ij}^{d*} x_{id}^{R}}, \quad j = 1, ..., n$$
(21)

$$\alpha_{j} = \frac{1}{n} \sum_{d=1}^{n} \frac{\sum_{j=1}^{s} \mu_{ij}^{d*} y_{d}^{L} - u_{s}^{d*}}{\sum_{j=1}^{m} \psi_{ij}^{d*} x_{id}^{L}}, \quad j = 1, ..., n$$

4. NUMERICAL EXAMPLE

To obtain the game cross efficiency according to proposed method, we consider five DMUs, with three inputs and two outputs. The inputs and outputs of DMUs are given in Table 1. To obtain the best cross efficiency, we evaluate DMUs using BCC model. Then To obtain the best cross efficiency, we evaluate DMUs using BCC model. Then using the obtained simple efficiencies, the best cross efficiencies are obtained by the proposed method. The results game cross efficiencies are shown in second row of the Table 2.

Table1 Inputs and outputs of DMUs

Table1. Inputs and outputs of DWOS						
		$\tilde{x_1}$	\tilde{x}_2	<i>x</i> ₃	Σ ₁	Ϋ́ 2
I	OMU1	(4,7,10)	(3,7,11)	(2,7,12)	(3,4,5)	(2,4,6)
I I	DMU2	(4,5,6)	(4,914)	(3,7,11)	(3,7,11)	(5,7,9)
	DMU3	(2,4,6)	(5,6,7)	(3,5,7)	(4,5,6)	(6,7,8)
Ι	DMU4	(2,5,8)	(7,9,11)	(5,8,11)	(5,6,7)	(1,2,3)
DMU5		(2,6,10)	(5,8,11)	(3,5,7)	(2,3,4)	(3,6,9)
	Table2. Game crosses efficiency of DMUs					
		DMU1	DMU2	DMU3	DMU4	DMU5
	Cross efficiency	0.60	0.98	0.99	0.81	0.72

5. CONCLUSION

This paper proposed a new DEA fuzzy game cross efficiency model to evaluate the best cross efficiency of DMUs under assumption of VRS for fuzzy data. The proposed method can be applied to the best cross efficiency under CRS assumption. For each DMU, a multiplier bundle is determined that minimizes the difference between simple efficiency and game cross efficiency. A large number of iterations should be done and take a lot of computation efforts, while the proposed model doesn't have this problem and multiplier bundle were obtained by solving only one model.

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