J. Basic. Appl. Sci. Res., **2(11)11164-11173, 2012**
© 2012, TextRoad Publication

ISSN 2090-4304
Journal of Basic and Applied Scientific Research
www.textroad.com

Ranking of Unit under Weight Restrictions Using Common Weights

Fatemeh Gholami Golsefid[a], Seyed Reza Seyed Nezhad Fahim[b]

[a]Department of Mathematics, Roudsar-Amlash Branch, Islamic Azad University, Roudsar, Iran
[b]Department of Accounting, Lahijan Branch, Islamic Azad University, Lahijan, Iran

ABSTRACT

Obviously, data envelopment analysis has not discrimination power to distinguish between DEA efficient units. In this paper, is firstly determined efficiency score of decision making units under weight restrictions (WR) model. This weight restrictions point importance of indexes (inputs and outputs). Next are proposed two methods to rank. In first method is determined one common set of weights for the performance indices of only efficient DMUs under weight restrictions. Then these DMUs are ranked according to the efficiency score weighted by the common set of weights. In second method an ideal line will be defined and determined a common set of weights for WR-efficient DMUs then a new efficiency score will be obtained and be ranked them with it.

KEYWORDS: DEA, Weight restrictions, Common weights, Efficiency score, Ideal line

1. INTRODUCTION

Charnes et al. [4] introduce data envelopment analysis (DEA) to assess the performances of a group of decision making units (DMUs) that utilize multiple inputs to produce multiple outputs. DEA divides DMUs successfully into two categories: efficient DMUs and inefficient DMUs. A ranking for inefficient DMUs is given, but the efficient DMUs have equal efficiency scores and they cannot be ranked.

Some of the methods which are proposed for ranking efficient DMUs are mentioned here. Anderson and Petersen [2] evaluate that DMUs efficiency possibly exceeds the conventional score 1.0, by comparing the DMU being evaluated with a linear combination of other DMUs, while excluding the observations of the DMU being evaluated. They try to discriminate between these efficient DMUs, by using different efficiency scores larger than 1.0. Cook et al. [7] developed prioritization models to rank only the efficient units in DEA. They divide those with equal scores, on the boundary, by imposing the restrictions on the multipliers (weights) in a DEA analysis. The idea of common weights in DEA was first introduced by Cook et al. [8] and Roll et al. [12] in the context of applying DEA to evaluate highway maintenance units. Cook and Kress [5, 6] gave a subjective ordinal preference ranking by developing common weights through a series of bounded DEA runs, by closing the gap between the upper and lower limits of the weights. Ganley and Cobbin [9] considered the common weights for all the units, by maximizing the sum of efficiency ratios of all the units, in order to rank each unit. They suggest the potential use of the common weights for ranking DMUs. Liu and Peng [11] searched common set weights to create the best efficiency score of one group composed of efficient DMUs. Then they use this common set of weights to evaluate the absolute efficiency of each efficient DMUs in order to rank them. Jahanshahloo et al. [10] proposed rank efficiency DMUs by comparing with an ideal line and the special line and obtain the common set of weights to evaluate the absolute efficient DMUs.

Meanwhile, the imposition of weight restrictions has been recognized as one of the important factors when applying DEA to actual situations and several models are developed for this purpose. These include the Assurance Region (AR) model by Thompson et al. [13] and the Con-ratio Approach by Charnes et al. [3]. Since weight for inputs (outputs) can be regarded as associated with costs of inputs and prices of outputs, constraints on weights should preferably reflect the actual costs (prices) information. In the AR model, upper and lower bounds are imposed on the ratio of weights for certain pairs of inputs or outputs. These weight restrictions contribute to avoiding the occurrence of frequently observed zero optimal weights to some inputs (outputs) that are caused by the optimization mechanism of DEA, and hence the results of analysis using weight restrictions are more persuasive than those without restrictions.

In this paper, firstly we will determine efficiency score of decision making units under weight restrictions (WR) model. This weight restrictions point importance of indexes (inputs and outputs). Next, we will propose two methods for ranking of WR- efficient units. In first method will be determined one common set of weights for the performance indices of only efficient DMUs under weight restrictions. Then these DMUs will be ranked according to the efficiency score weighted by the common set of weights. In second method an ideal line will be defined and

*Corresponding Author: Fatemeh Gholami Golsefid, Department of Mathematics, Roudsar-Amlash Branch, Islamic Azad University, Roudsar, Iran. **Tel/fax**: +981413222284 **E-mail**: F.gholami@iauroudsar-amlash.ac.ir

determined a common set of weights for efficient DMUs then a new efficiency score will be obtained and ranked them with it.

2. The weight restriction model

In this section, a more general model (called the "weight restriction" model) introduced. Assume, we have n DMUs that are evaluated in terms of m inputs and s outputs. Let x_{ij} and y_{rj} be input and output values of DMU$_j$ for $i = 1,2,\dots,m$ and $r = 1,2,\dots,s$. Spot CCR model, the efficiencies of the n DMUs using weight restrictions are measured by the following model.

$$\begin{aligned}
\theta_W^* = max \quad & \sum_{r=1}^{s} u_r y_{rp} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ip} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j = 1,2,\dots,n \\
& \sum_{i=1}^{m} v_i p_{ik} \le 0 \qquad k = 1,2,\dots,(2m-2) \\
& \sum_{r=1}^{s} u_r q_{rt} \le 0 \qquad t = 1,2,\dots,(2s-2) \\
& v_i \ge 0 \qquad i = 1,2,\dots,m \\
& u_r \ge 0 \qquad r = 1,2,\dots,s
\end{aligned} \tag{1}$$

Where $P_{m\times 2m-2} = (p_{ik})$ and $Q_{s\times 2s-2} = (q_{rt})$ are matrices that are associated with weight restrictions as described below. For example, if ratio of weights for initial and i th of input and initial and r th of output be as follows:

$$l_{1i} \le v_i/v_1 \le u_{1i} \Rightarrow l_{1i}v_1 \le v_i \le u_{1i}v_1 \qquad i = 2,3,\dots,m$$
$$L_{1r} \le u_r/u_1 \le U_{1r} \Rightarrow L_{1r}u_1 \le u_r \le U_{1r}u_1 \qquad r = 2,3,\dots,s$$

Where l_{1i} and u_{1i} are lower and upper bound of v_i/v_1, and L_{1r} and U_{1r} are lower and upper bound of u_r/u_1.In this case the matrices P and Q are defined as follows:

$$P = \begin{bmatrix} l_{12} & -u_{12} & l_{13} & -u_{13} & \cdots & \cdots \\ -1 & 1 & 0 & 0 & \vdots & \vdots \\ 0 & 0 & -1 & 1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \end{bmatrix}_{m\times 2m-2}$$

$$Q = \begin{bmatrix} L_{12} & -U_{12} & L_{13} & -U_{13} & \cdots & \cdots \\ -1 & 1 & 0 & 0 & \vdots & \vdots \\ 0 & 0 & -1 & 1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \end{bmatrix}_{s\times 2s-2}$$

This is clear, that if weight restrictions of inputs be as $v_1 l_{12} \le v_2 \le v_1 u_{12}$ and $v_2 l_{23} \le v_3 \le v_2 u_{23}$ and …, then the matrices P will change. Also it is indefeasible for Q matrices.

The dual of (1) model is as follows:

$$\begin{aligned}
\theta_W^* = \min \quad & \theta_w \\
s.t. \quad & \theta_w x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{k=1}^{2m-2} p_{ik}\pi_k - s_i^- = 0 \qquad i = 1,2,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{2s-2} q_{rt}\tau_t - s_r^+ = y_{rp} \qquad r = 1,2,\dots,s \\
& \lambda_j \ge 0 \qquad j = 1,2,\dots,n \\
& \pi_k \ge 0 \qquad k = 1,2\dots,2m-2 \\
& \tau_t \ge 0 \qquad t = 1,2,\dots,2s-2 \\
& s_i^- \ge 0 \qquad i = 1,2,\dots,m \\
& s_r^+ \ge 0 \qquad r = 1,2,\dots,s
\end{aligned} \tag{2}$$

Definition 1(WR- efficiency): The DMU$_p$ is WR- efficient if and only if, for optimum solutions of (2) model $(\theta_w^*, \lambda^*, \pi^*, \tau^*, S^{-*}, S^{+*})$ it satisfies:

$$\theta_w^* = 1 \quad , \quad S^{-*} = 0 \quad , \quad S^{+*} = 0$$

Or The DMU$_p$ is WR- efficient if and only if, for optimum solutions of model (1), (v^*, u^*) it satisfies:

$$u^* y_p = 1 \quad , \quad v^* > 0 \quad , \quad u^* > 0$$

In otherwise DMU$_p$ is WR- inefficient.

3. Ranking by common weights under weight restrictions (CW under WR)

Suppose reference set E be all DMUs of WR-efficient. Decision makers always intuitively take the maximal efficiency score 1.0 as the common benchmark level for DMUs. In this case, weighted sum of inputs is equal weighted sum of outputs. By the definition of the efficiency score, the common benchmark level is one straight line that passes through the origin, with slope 1.0 in the coordinate. Object is giving or approximating of DMUs into benchmark. In Fig. 1 the vertical and horizontal axes are set to be the virtual output (weighted sum of s outputs) and virtual input (weighted sum m inputs), respectively. For any two DMUs, DMU_M and DMU_N, if given one set of weights u_r' $(r = 1,2,\dots,s)$ and v_i' $(i = 1,2,\dots,m)$, then the coordinate of points M' and N' in Fig. 1 are $\left(\sum_{i=1}^{m} v_i' x_{iM}, \sum_{r=1}^{s} u_r' y_{rM}\right)$ and $\left(\sum_{i=1}^{m} v_i' x_{iN}, \sum_{r=1}^{s} u_r' y_{rN}\right)$. The notation of a decision variable with superscript symbols" ' " represents an arbitrary assigned value. The virtual distances, between points M' and M'^{P} on the horizontal axes and vertical axes, are denoted as $\Delta_{M'}^{I}$ and $\Delta_{M'}^{O}$, respectively. Similarly, for points N' and N'^{P}, the distances are $\Delta_{N'}^{I}$ and $\Delta_{N'}^{O}$. Therefore, in view of points M' and N', we observe that exists a total virtual distance $\Delta_{M'}^{I} + \Delta_{M'}^{O} + \Delta_{N'}^{I} + \Delta_{N'}^{O}$ to the benchmark line.

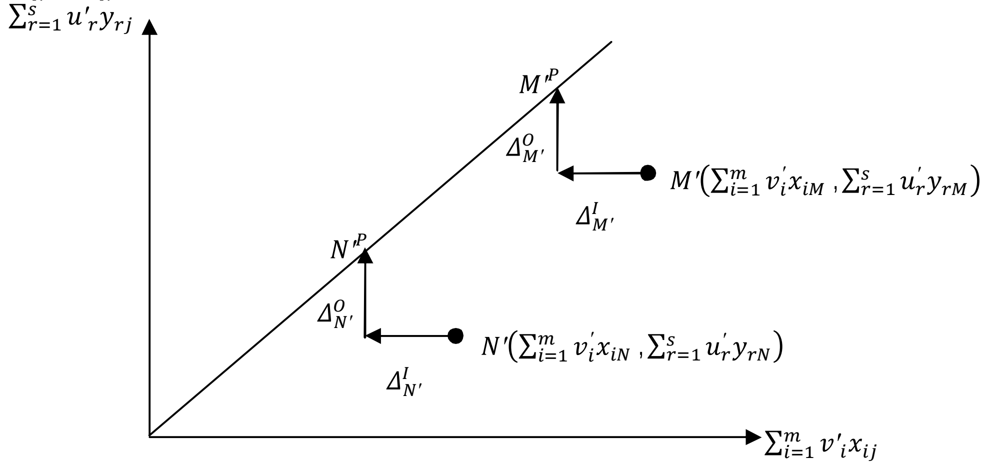

Fig. 1. Distance analysis showing DMU below the virtual benchmark line.

If we represent the optimal value of the variable with superscript "$\star$", we should determine an optimal set of weights $u_r^{\star}$ $(r = 1,2,\dots,s)$ and $v_i^{\star}$ $(i = 1,2,\dots,m)$ such that both points $M^{\star}$ and $N^{\star}$ below the benchmark line could be as close to their projection points, $M^{\star P}$ and $N^{\star P}$ on the benchmark line, as possible. In other words, by adopting the optimal weights, the total virtual gap $\Delta_{M^{\star}}^{I} + \Delta_{M^{\star}}^{O} + \Delta_{N^{\star}}^{I} + \Delta_{N^{\star}}^{O}$ to the benchmark line is the shortest to both DMUs. The object function is minimizing of general distance of DMUs, inset E, to benchmark line. As for the constraint, the numerator is the weighted sum of outputs plus the vertical virtual distance Δ_j^{O} and the denominator is the weighted sum of inputs minus the horizontal virtual distance Δ_j^{I}. The constraint implies that the direction closest to the benchmark line is upwards and leftwards at same time. The ratio of the numerator to the denominator equals to 1.0, which means that the projection point on the benchmark line is reached. In this method, it assumed that the benchmark line is located above all DMUs in set E. The optimal common set of weights $u_r^{\star}$ $(r = 1,2,\dots,s)$ and $v_i^{\star}$ $(i = 1,2,\dots,m)$ to each WR- efficient DMU would be solved and then each WR- efficient DMU could obtain one absolute efficiency score as the standard for comparison. Then, ranking of those WR- efficient DMUs would be completed. The basic difference of this model and method of Liu and Peng [11] is weight restrictions applied.

$$
\begin{aligned}
\Delta^{\star} = \min \quad & \sum_{j\in E}\left(\Delta_j^{I} + \Delta_j^{O}\right) & & \\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + \Delta_j^{O}}{\sum_{i=1}^{m} v_i x_{ij} - \Delta_j^{I}} = 1 & & j \in E \\
& \sum_{i=1}^{m} v_i p_{ik} \le 0 & & k = 1,2,\dots,(2m-2) \\
& \sum_{r=1}^{s} u_r q_{rt} \le 0 & & t = 1,2,\dots,(2s-2) \qquad (3) \\
& \Delta_j^{O}, \Delta_j^{I} \ge 0 & & j \in E \\
& v_i \ge \varepsilon > 0 & & i = 1,2,\dots,m \\
& u_r \ge \varepsilon > 0 & & r = 1,2,\dots,s
\end{aligned}
$$

The ε is a positive Archimedean infinitesimal constant, and $P_{m\times 2m-2} = (p_{ik})$ and $Q_{s\times 2s-2} = (q_{rt})$ are matrices that, are depended weight restrictions. The ratio form of constraints in (3) model can be rewritten in a linear form, formulated in the constraints of (4) model.

$$\begin{array}{lll} \Delta^{\star} = \min & \sum_{j\in E}\left(\Delta_j^I + \Delta_j^O\right) & \\ \quad s.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \left(\Delta_j^O + \Delta_j^I\right) = 0 & j \in E \\ & \sum_{i=1}^{m} v_i p_{ik} \le 0 & k = 1,2,\dots,(2m-2) \\ & \sum_{r=1}^{s} u_r q_{rt} \le 0 & t = 1,2,\dots,(2s-2) \\ & \Delta_j^O, \Delta_j^I \ge 0 & j \in E \\ & v_i \ge \varepsilon > 0 & i = 1,2,\dots,m \\ & u_r \ge \varepsilon > 0 & r = 1,2,\dots,s \end{array} \tag{4}$$

Then, if we let $\Delta_j = \Delta_j^I + \Delta_j^O$, (4) model simplified to the following linear programming.

$$\begin{array}{lll} \Delta^{\star} = \min & \sum_{j\in E}\Delta_j & \\ \quad s.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \Delta_j = 0 & j \in E \\ & \sum_{i=1}^{m} v_i p_{ik} \le 0 & k = 1,2,\dots,(2m-2) \\ & \sum_{r=1}^{s} u_r q_{rt} \le 0 & t = 1,2,\dots,(2s-2) \\ & \Delta_j \ge 0 & j \in E \\ & v_i \ge \varepsilon > 0 & i = 1,2,\dots,m \\ & u_r \ge \varepsilon > 0 & r = 1,2,\dots,s \end{array} \tag{5}$$

This problem be rewritten to the equivalent linear programming (6) by taking out the slack variable Δ_j and putting $X_i = \sum_{j\in E} x_{ij}$, $i = 1,2,\dots,m$ and $Y_r = \sum_{j\in E} y_{rj}$, $r = 1,2,\dots,s$ then $\Delta_j = -(\sum_{r=1}^{s} u_r Y_r - \sum_{i=1}^{m} v_i X_i)$.

$$\begin{array}{lll} -\Delta^{\star} = \max & (\sum_{r=1}^{s} u_r Y_r - \sum_{i=1}^{m} v_i X_i) & \\ \quad s.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 & j \in E \\ & \sum_{i=1}^{m} v_i p_{ik} \le 0 & k = 1,2,\dots,(2m-2) \\ & \sum_{r=1}^{s} u_r q_{rt} \le 0 & t = 1,2,\dots,(2s-2) \\ & v_i \ge \varepsilon > 0 & i = 1,2,\dots,m \\ & u_r \ge \varepsilon > 0 & r = 1,2,\dots,s \end{array} \tag{6}$$

The difference is that (6) is used to search one common set of weights, in order to evaluate the absolute efficiency score. Moreover, (6) can be used to discriminate between the DEA efficient DMUs which resulted from the DEA model in the CRS case. In order to obtain more information, we transform (6) to its dual form (7).

$$\begin{array}{lll} \max & \varepsilon(\sum_{i=1}^{m} \alpha_i + \sum_{r=1}^{s} \beta_r) & \\ s.t. & \sum_{j\in E} \lambda_j x_{ij} + \sum_{k=1}^{2m-2} \pi_k p_{ik} + \alpha_i = X_i & i = 1,2,\dots,m \\ & \sum_{j\in E} \lambda_j y_{rj} + \sum_{t=1}^{2s-2} \tau_t q_{rt} - \beta_r = Y_r & r = 1,2,\dots,s \\ & \lambda_j \ge 0 \quad j = 1,2,\dots,n & \\ & \pi_k \ge 0 \quad k = 1,2\dots,2m-2 & \\ & \tau_t \ge 0 \quad t = 1,2,\dots,2s-2 & \\ & \alpha_i \ge 0 \quad i = 1,2,\dots,m & \\ & \beta_r \ge 0 \quad r = 1,2,\dots,s & \end{array} \tag{7}$$

Definition 2: By optimal weights $v_i^{\star}$ and $u_r^{\star}$, the CW-efficiency score under WR of DMU_j is defined as follows:

$$\xi_j^{\star} = \frac{\sum_{r=1}^{s} u_r^{\star} y_{rj}}{\sum_{i=1}^{m} v_i^{\star} x_{ij}} \qquad j \in E$$

By the value of efficiency score, we can distinguish the DMUs into two separable group, DMUs of CW-efficient under WR (on the benchmark line) and DMUs of CW-inefficient under WR (below the benchmark line).

Definition 3:

a. If $\Delta_j^{\star} = 0$ or $\xi_j^{\star} = 1$, then DMU_j is CW-efficient under WR (on the benchmark line). Otherwise, DMU_j is CW-inefficient under WR (below the benchmark line).
b. If $\xi_j^{\star} > \xi_i^{\star}$, then the performance of DMU_j is better than DMU_i.

c. If $\xi_j^\star = \xi_i^\star < 1$, i.e. both DMUs are CW-inefficient under WR (below benchmark line), then the performance of DMU_j is better than DMU_i if $\Delta_j^\star < \Delta_i^\star$.
d. If $\xi_j^\star = \xi_i^\star = 1$, i.e. both DMUs are CW-efficient under WR (on the benchmark line), then the performance of DMU_j is better than DMU_i if $\lambda_j^\star > \lambda_i^\star$.

The CW-efficiency score under WR of DMU_j , (j∈E) is limited to no greater than 1.0, therefore there is no DMU upon the benchmark line, and there is at least one DMU that joins the assessment located on the benchmark line.

4. Common set of weights under WR by comparing with the ideal line

In this section, in the first, we define ideal DMU and rank WR- efficient DMUs with common weights by ideal line.

Definition 4 (ideal DMU): The ideal DMU is a DMU, that its inputs are minimize inputs of all of DMUs and its outputs is maximize outputs of all of DMUs. In other word, if we show ideal DMU by $\overline{DMU} = (\bar{x}, \bar{y})$, then $\bar{x}_i = \min_{j=1,\dots,n} x_{ij}$, $(i = 1, \dots, m)$ and $\bar{y}_r = \min_{j=1,\dots,n} y_{rj}$, $(r = 1, \dots, s)$. Meanwhile, an ideal line is one straight line that passes through the origin and ideal DMU with slope 1.0.

In space of weighted sum of inputs and weighted sum of outputs, ox is an ideal line and $\overline{DMU} = (\sum_{i=1}^{m} v'_i \bar{x}_i, \sum_{r=1}^{s} u'_r \bar{y}_r)$ is an ideal DMU. The notation of a decision variable with superscript symbols" ' " represents an arbitrary assigned value. For any two DMUs, DMU_M and DMU_N, if given one set of weights u'_r $(r = 1,2, \dots, s)$ and v'_i $(i = 1,2, \dots, m)$, then the coordinate of points M' and N' in Fig. 2 are $\left(\sum_{i=1}^{m} v'_i x_{iM} , \sum_{r=1}^{s} u'_r y_{rM}\right)$ and $\left(\sum_{i=1}^{m} v'_i x_{iN} , \sum_{r=1}^{s} u'_r y_{rN}\right)$. The virtual distances, between points M' and M'^P on the horizontal axes and vertical axes, are denoted as $\Delta_{M'}^I$ and $\Delta_{M'}^O$, respectively. Similarly, for points N' and N'^P, the distances are $\Delta_{N'}^I$ and $\Delta_{N'}^O$. Therefore, in view of points M' and N', we observe that exists a total virtual distance $\Delta_{M'}^I + \Delta_{M'}^O + \Delta_{N'}^I + \Delta_{N'}^O$ to the ideal line.

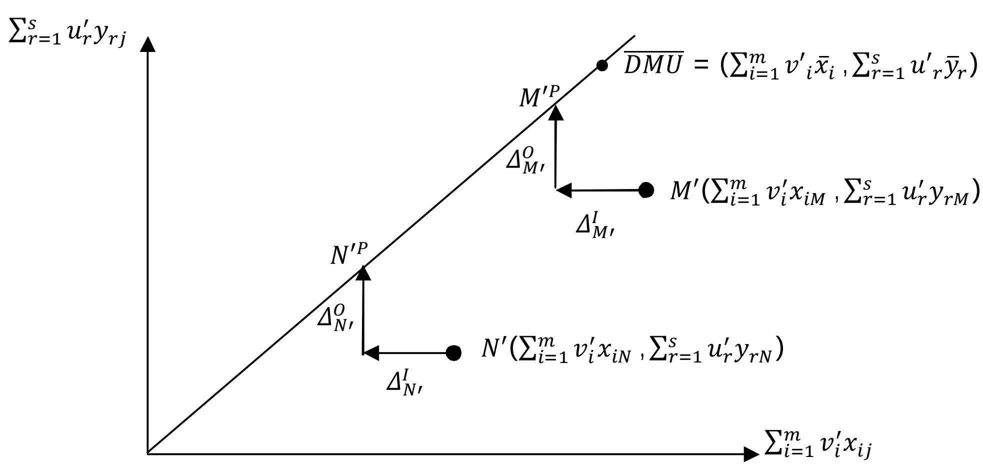

Fig. 2. Distance analysis showing DMU below the virtual ideal line.

If we represent the optimal value of the variable with superscript "$\star$", we should determine an optimal set of weights $u_r^\star$ $(r = 1,2, \dots, s)$ and $v_i^\star$ $(i = 1,2, \dots, m)$ such that both points $M^\star$ and $N^\star$ below the ideal line could be as close to their projection points, $M^{\star P}$ and $N^{\star P}$ on the ideal line, as possible. In other words, by adopting the optimal weights, the total virtual gap $\Delta_{M^\star}^I + \Delta_{M^\star}^O + \Delta_{N^\star}^I + \Delta_{N^\star}^O$ to the ideal line is the shortest to both DMUs.

The object function is minimizing of general distance of DMUs, inset E, to ideal line. As for the constraint, the numerator is the weighted sum of outputs plus the vertical virtual distance Δ_j^O and the denominator is the weighted sum of inputs minus the horizontal virtual distance Δ_j^I. The constraint implies that the direction closest to the ideal line is upwards and leftwards at same time. The ratio of the numerator to the denominator equals to 1.0, which means that the projection point on the ideal line is reached. The basic difference of this model and method of Jahanshahloo et al. [10] is weight restrictions applied. Then we purpose following model.

$$
\begin{aligned}
\Delta^{\star} = \min \quad & \textstyle\sum_{j\in E}\left(\Delta_j^I + \Delta_j^O\right) \\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r \bar{y}_r}{\sum_{i=1}^{m} v_i \bar{x}_i} = 1 \\
& \frac{\sum_{r=1}^{s} u_r y_{rj} + \Delta_j^O}{\sum_{i=1}^{m} v_i x_{ij} - \Delta_j^I} = 1 && j \in E \qquad (8)\\
& \textstyle\sum_{i=1}^{m} v_i p_{ik} \le 0 && k = 1,2,\dots,(2m-2) \\
& \textstyle\sum_{r=1}^{s} u_r q_{rt} \le 0 && t = 1,2,\dots,(2s-2) \\
& \Delta_j^O, \Delta_j^I \ge 0 && j \in E \\
& v_i \ge \varepsilon > 0 && i = 1,2,\dots,m \\
& u_r \ge \varepsilon > 0 && r = 1,2,\dots,s
\end{aligned}
$$

The ratio form of constraints in (8) model can be rewritten in a linear form, formulated in the constraints of following model.

$$
\begin{aligned}
\Delta^{\star} = \min \quad & \textstyle\sum_{j\in E}\left(\Delta_j^I + \Delta_j^O\right) \\
s.t. \quad & \textstyle\sum_{r=1}^{s} u_r \bar{y}_r - \sum_{i=1}^{m} v_i \bar{x}_i = 0 \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \left(\Delta_j^O + \Delta_j^I\right) = 0 && j \in E \\
& \textstyle\sum_{i=1}^{m} v_i p_{ik} \le 0 && k = 1,2,\dots,(2m-2) \\
& \textstyle\sum_{r=1}^{s} u_r q_{rt} \le 0 && t = 1,2,\dots,(2s-2) \qquad (9)\\
& \Delta_j^O, \Delta_j^I \ge 0 && j \in E \\
& v_i \ge \varepsilon > 0 && i = 1,2,\dots,m \\
& u_r \ge \varepsilon > 0 && r = 1,2,\dots,s
\end{aligned}
$$

Then, if we let $\Delta_j = \Delta_j^I + \Delta_j^O$, (9) model simplified to the following linear programming.

$$
\begin{aligned}
\Delta^{\star} = \min \quad & \textstyle\sum_{j\in E} \Delta_j \\
s.t. \quad & \textstyle\sum_{r=1}^{s} u_r \bar{y}_r - \sum_{i=1}^{m} v_i \bar{x}_i = 0 && (\ast) \\
& \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \Delta_j = 0 && j \in E \\
& \textstyle\sum_{i=1}^{m} v_i p_{ik} \le 0 && k = 1,2,\dots,(2m-2) \\
& \textstyle\sum_{r=1}^{s} u_r q_{rt} \le 0 && t = 1,2,\dots,(2s-2) \qquad (10)\\
& \Delta_j \ge 0 && j \in E \\
& v_i \ge \varepsilon > 0 && i = 1,2,\dots,m \\
& u_r \ge \varepsilon > 0 && r = 1,2,\dots,s
\end{aligned}
$$

If a DMU_j was on ideal line, then we use definition (2) for CW- efficiency score under WR. Therefore the CW-efficiency score under WR of it is 1.0. So that constrain ($\ast$) in (10) become redundant and this model become same model of (5). On the other hand, the ideal line is the benchmark line. We result (5) is special case of (10). Therefore, DMU_j is CW- efficient under WR if $\Delta_j^{\star} = 0$ or $\xi_j^{\star} = 1$, otherwise, DMU_j is CW- inefficient under WR.

Definition 5: The performance of DMU_j is better than DMU_i if $\Delta_j^{\star} < \Delta_i^{\star}$.

5. Numerical example

In this section, we apply the above methods for two numerical examples.

Example 1: We consider the following example from Tone [14] paper. The data set are inclusive 16 hospitals. As for the inputs and outputs, we employ the following technical factors as designated in Table 1.

Input

- Doctor: Total hours worked by doctors in the survey period
- Nurse: Total hours worked by nurse
- Tech: Total hours worked by technical workers
- Office: Total hours worked by office staff

Output

- Outpatient: Total medical insurance points for outpatients
- Inpatient: Total medical insurance points for inpatients

Among the inputs we chose "Doctor" as axis and impose weight restriction of other input relative to "Doctor" as below where v_1, v_2, v_3 and v_4 represent the weights for Doctor, Nurse, Technical worker and Office staff, respectively. The lower / upper bounds were determined by considering the ratio of cost of inputs against doctor's cost.

$$2 \le {}^{v_1}/_{v_2} \le 5.3$$
$$1.7 \le {}^{v_1}/_{v_3} \le 4.2$$
$$2.3 \le {}^{v_1}/_{v_4} \le 5.4$$

In the similar way, we added the weight restriction outputs as:

$$0.28 \le {}^{u_1}/_{u_2} \le 0.4$$

Where u_1 and u_2 represent the weights for outpatient and inpatient, respectively. The results performance obtained by the addition of these weight restrictions are exhibited in Table 1.

Table 1. Technical data for 16 hospitals

DMU	I_1= Doctor	I_2= Nurse	I_3=Tech	I_4=Office	O_1=Outpat	O_2=Inpat	$\theta_W^\star$
H1	995	6205	1375	2629	4127	1678	1.0000
H2	917	5898	1379	2047	3721	1277	0.9230
H3	3178	10049	3615	3511	2706	2051	0.5212
H4	813	5833	1124	1730	2176	1538	0.9097
H5	1236	8639	2486	4990	5220	2042	0.8618
H6	1146	7610	1600	3589	3517	1856	0.8283
H7	705	5600	1557	3623	2352	2060	1.0000
H8	2871	11524	2880	2452	1755	1664	0.4078
H9	1098	8998	1730	2823	4412	2334	1.0000
H10	2032	9383	2421	4454	5386	2080	0.7690
H11	1414	10468	2140	3649	5735	2691	0.9865
H12	1967	11260	2759	3178	6079	2804	0.9246
H13	1851	9880	2335	4570	5893	2495	0.8751
H14	3100	15649	5487	2940	5248	3692	0.7545
H15	5016	18010	4008	3567	7800	4582	0.8056
H16	1924	12682	2490	2975	6040	3396	1.0000

In Table 1, it is shown that hospitals H1, H7, H9 and H5 are the WR-efficient hospitals. The WR-inefficient hospitals are ranked by $\theta_W^\star$ easily. For ranking of WR-efficient hospitals, we apply (5) and (7) models (Table 2). Then, we introduce ideal hospital and compare them with ideal hospital. Finally, by (10) model we rank WR-efficient hospitals (Table 3).

Table 2. Results ranking by comparing with benchmark line

Hospital	$\xi_j^\star$	$\Delta_j^\star$	$\lambda_j^\star$	Rank
H1	1.0000	—	2.4667	1
H7	0.9840	0.2470	—	4
H9	1.0000	—	1.8260	2
H16	1.0000	—	0.1379	3

The ideal hospital is as following:

I_1=705 I_2=5600 I_3=1124 I_4=1730 O_1=7800 O_2=4582

Table 3. Results ranking by comparing with ideal line

Hospital	$\Delta_j^\star$	Rank
H1	14.8620	2
H7	14.1249	1
H9	20.4892	3
H16	29.5925	4

For ranking by comparing with benchmark line Table 2 shows: $\xi_1^\star = \xi_9^\star = \xi_{16}^\star = 1 > \xi_7^\star = 0.9840$ and $\lambda_1^\star > \lambda_9^\star > \lambda_{16}^\star$. By Definition 3, we can say that, performance H1 is better than H9, and its performance is better than H16, and its performance is better than H7.

For ranking by comparing with ideal line Table 3 shows: $\Delta_7^\star < \Delta_1^\star < \Delta_9^\star < \Delta_{16}^\star$. By Definition 5, we can say that, performance H7 is better than H1, and its performance is better than H9, and its performance is better than H16. Finally, Table 4 shows ranking of all of DMUs (hospitals) in two methods.

Table 4. Results ranking all of DMUs in two methods

Hospital	Ranking by comparing with benchmark line	Ranking by comparing with ideal line
H1	1	2
H2	7	7
H3	15	15
H4	8	8
H5	10	10
H6	11	11
H7	4	1
H8	16	16
H9	2	3
H10	13	13
H11	5	5
H12	6	6
H13	9	9
H14	14	14
H15	12	12
H16	3	4

Example2. In this example, we consider 20 branches of bank in Iran that previously analyzed by Amirteimoori and Kordrostami [1] and Jahanshahloo et al. [10] and are listed in Table 5.

Table 5. Data of 20 branches of bank

Branch	I_1 = Staff	I_2 = Computer	I_3 = Space	O_1 = Deposits	O_2 = Loans	O_3 = Change	$\theta_W^\star$
B1	0.950	0.700	0.155	0.190	0.521	0.293	0.6983
B2	0.796	0.600	1.000	0.227	0.627	0.462	0.7222
B3	0.798	0.750	0.513	0.228	0.970	0.261	0.9091
B4	0.865	0.550	0.210	0.193	0.632	1.000	1.0000
B5	0.815	0.850	0.268	0.233	0.722	0.246	0.8387
B6	0.842	0.650	0.500	0.207	0.603	0.569	0.7170
B7	0.719	0.600	0.350	0.182	0.900	0.716	1.0000
B8	0.785	0.750	0.120	0.125	0.234	0.298	0.4758
B9	0.476	0.600	0.135	0.080	0.364	0.244	0.7074
B10	0.678	0.550	0.510	0.082	0.184	0.049	0.2515
B11	0.711	1.000	0.305	0.212	0.318	0.403	0.5502
B12	0.811	0.650	0.255	0.123	0.923	0.628	1.0000
B13	0.659	0.850	0.340	0.176	0.645	0.261	0.7351
B14	0.976	0.800	0.540	0.144	0.514	0.243	0.4357
B15	0.685	0.950	0.450	1.000	0.262	0.098	1.0000
B16	0.613	0.900	0.525	0.115	0.402	0.464	0.5847
B17	1.000	0.600	0.205	0.090	1.000	0.161	0.9849
B18	0.634	0.650	0.235	0.059	0.349	0.068	0.4380
B19	0.372	0.700	0.238	0.039	0.190	0.111	0.3324
B20	0.583	0.550	0.500	0.110	0.615	0.764	1.0000

Among the inputs we chose “Staff” as axis and impose weight restriction of other input relative to “Staff” as below where v_1, v_2 and v_3 represent the weights for Staff, Computer, and Space, respectively.

$$0.6 \leq {v_1}/{v_2} \leq 4.4$$
$$0.5 \leq {v_1}/{v_3} \leq 5.5$$

In the similar way, we added the weight restriction outputs as:

$$0.7 \leq {u_1}/{u_2} \leq 5.4$$
$$0.85 \leq {u_1}/{u_3} \leq 6.8$$

Where u_1, u_2 and u_3 represent the weights for Deposits, Loans and Change, respectively.

The results performance obtained by the addition of these weight restrictions are exhibited in Table 5. In this Table, it is shown that branches of banks B4, B7, B12, B15 and H20 are the WR-efficient branches. For ranking of WR-efficient branches, we apply all of methods of example 1.Table 6 and 7 show results.

Table 6. Results ranking of branches of bank by comparing with benchmark line

Hospital	$\xi_j^\star$	$\Delta_j^\star$	$\lambda_j^\star$	Rank
B4	1.0000	—	1.6247	2
B7	1.0000	—	2.3067	1
B12	0.8688	0.0003	—	4
B15	1.0000	—	0.8746	3
B20	0.8477	0.0003	—	5

The ideal hospital is as following:

I_1=0.372 I_2=0.550 I_3=0.120 O_1=1.000 O_2=1.000 O_3=1.000

Table 7. Results ranking of branches of bank by comparing with ideal line

Hospital	$\Delta_j^\star$	Rank
B4	0.002367	2
B7	0.002394	3
B12	0.002815	4
B15	0.004159	5
B20	0.002363	1

For ranking by comparing with benchmark line Table 6 shows: $\xi_4^\star = \xi_7^\star = \xi_{15}^\star = 1 > \xi_{12}^\star > \xi_{20}^\star$ and$\Delta_{12}^\star = \Delta_{20}^\star$ and $\lambda_7^\star > \lambda_4^\star > \lambda_{15}^\star$. By Definition 3, we can say that, performance B7 is better than B4, and its performance is better than B15, and its performance is better than B12, and its performance is better than B20.

For ranking by comparing with ideal line Table 7 shows: $\Delta_{20}^\star < \Delta_4^\star < \Delta_7^\star < \Delta_{12}^\star < \Delta_{15}^\star$. By Definition 5, we can say that, performance B20 is better than B4, and its performance is better than B7, and its performance is better than B12, and its performance is better than B15.

6. Conclusion

Ranking of DMUs in DEA is an important phase for efficiency evaluation of DMUs. DEA techniques generally do not rank the efficient DMUs. In this paper, we impose weight restrictions in CCR model. This weight restrictions point importance of indexes (inputs and outputs). Then, we determine efficiency score of decision making units under weight restrictions (WR) model. Next, we propose two methods to rank. In first method we determine one common set of weights for the performance indices of only efficient DMUs under weight restrictions. Then, we rank these DMUs according to the efficiency score weighted by the common set of weights. In second method we define an ideal line and determine a common set of weights for WR-efficient DMUs then a new efficiency score obtain and rank them with it.

REFERENCES

1. Amirteimoori, A. R. and S. Kordrostami, 2005. Efficient surfaces and an efficiency index in DEA: A constant returns to scale. Applied Mathematics and Computation, 163, 683691.
2. Anderson, P. and N. C. Petersen, 1993. A procedure for ranking efficient units in data envelopment analysis. Management Science, 39(10): 1261-1264
3. Charnes, A., W. W. Cooper, Z. M. Huang and D. B. Sun, 1990. Polyhedral Cone-Ratio DEA models with an illustrative application to large commercial banks. Journal of Econometrics, 46, 73-91.
4. Charnes, A., W. W. Cooper and E. Rhodes, 1978. Measuring the efficiency of decision making units . European Journal of Operational Research, 2, 42944.
5. Cook, W. and M. Kress, 1990. Data envelopment model for aggregating preference ranking. Management Science, 36(11), 1302-10.
6. Cook, W. and M. Kress, 1991. A multiple criteria decision model with ordinal preference data. European Journal of Operational Research, 54, 191-8.
7. Cook, W., M. Kress and L. M. Seiford, 1992. Prioritization models for frontier decision making units in DEA. European Journal of Operational Research, 59, 319-323.
8. Cook, W., Y. Roll and A. Kazakov, 1990. ADEA model for measuring the relative efficiencies of highway maintenance patrols. INFOR, 28(2), 113-24.
9. Ganley, J. and J. Cubbin, 1992. Public sector efficiency measurement: applications of data envelopment analysis. Amsterdam: North-Holland.
10. Jahanshahloo, G. R., F. Hosseinzadeh lotfi, M. Khanmohammadi, M. Kazemimanesh and V. Rezaie, 2010. Ranking of units by positive ideal DMU with common weights. Expert Systems with Applications, 37, 7483-7488.
11. Liu, F. F. and H. H. Peng, 2006. Ranking of units on the DEA frontier with common weights. Computers and Operations Research.doi:10.1016/j.cor.2006.09.006.
12. Roll, Y., W. Cook and B. Golany, 1991. Controlling factor weights in data envelopment analysis. IIE Transactions, 24, 1-9.
13. Tompson, R. G., F. D. Singleton Jr, R. M. Thrall and B. A. Smith, 1986. Comparative site evaluations for locating a high-energy physics lab in Texas. Interface, 16, 35-49.
14. Tone, K. 2001. On returns to scale under weight restrictions in data envelopment analysis. Journal of Productivity Analysis, 16, 31-47.