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Investigation of Elastoplastic Functionally Graded Euler-Bernoulli Beam Subjected to Distributed Transverse Loading

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ABSTRACT

In this paper, an elastoplastic FGM simply supported Euler-Bernoulli beam with rectangular section subjected to uniformly distributed transverse loading has been investigated by variational method. Material properties define by power law. The poisson's ratio of the beam assumed to be constant. But the young's modules vary continuously throughout the thickness direction symmetrically. The analytical solution illustrates stress response of the beam and the required moment to have fully plastic beam is determined.
KEY WORDS: Euler-Bernoulli Beam, FGM, Energy Method, transverse loading, elastoplastic

4. INTRODUCTION

Functionally graded materials (FGMs) are microscopically inhomogeneous composite that are usually made from a mixture of metals and ceramics. FGM are regarded as one of the most promising candidates for future advancedcomposites in many engineering sector such as the aerospace, aircraft, automobile, and defense industries, and most recently the electronic and bio medical sector [1].

Traditional composites comprised of two different materials have been widely used to satisfy the high performance demands. However, stress singularities in such composites may occur at the interface between two different materials,due to themismatch ofmaterials. Especially, in a high-temperature environment, for example in the engine combustion chamber of an air vehicle or a nuclear fusion reaction container, the relatively higher mismatch in thermal expansion coefficients will inducehigh residual stresses. Consequently, the composite may incur cracking or deboning. Therefore, the concept of such material (FGM) was introduced to satisfy the demand of ultra-high-temperature environment and to eliminate the stress singularities [2,3].

An FGM can be prepared by continuously changing the constituents of multi-phase materials in a pre-determined volume fraction of the constituent material [4–6]. Due to the continuous change in material properties of an FGM, the interfaces between two materials disappear but the characteristics of two or more different materials of the composite are preserved. Studies reveal that the thermal residual stresses can be significantly relaxed by using a FGM [7–8].

Power-law function [9–10] and exponential function [11–13] are commonly used to describe the variations of material properties of FGMs. However, in both power-law and exponential functions, the stress concentrations appear in one of the interfaces in which the material is continuous but rapidly changing. Therefore, Hung and Chi[8] proposed a sigmoid FGM, which is composed of two power-law functions to define a new volume fraction. Chiand Chung [14] indicated that the use of a sigmoid FGM can significantly reduce the stress intensity factors of a cracked body. Because of the wide material variations and applications of FGMs, literatures corresponding to FGMs in the material constituent [14–15], fracture mechanics [9,12] have been rapidly increased in the last 10 years. The FGM may be applied to plate and beam structures as a thermal barrier. Therefore, understanding the mechanical behavior of an FGM beam is very important to assess the safety of the beam structure. To design a beam, it is necessary to have information about stress and maximum moment to have fully plastic beam.

The literature reveals a continued interest among the research community to develop efficient mathematical models to predict the static response of thin and thick beams. In this paper, the principle of virtual work (PVW) is used to obtain the static equilibrium equations and the boundary conditions for functionally graded beam with distributed transverse loading. Deflection, stress distribution and the required moment to have fully plastic beam are presented.

2. ANALYSIS

In FGM material, the properties define under a specific function. Here, elasticity module and yield stress define as follow;

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$$E = E_i\left(\left|\frac{y}{h}\right| + \frac{1}{2}\right)^n \quad (1)$$

$$Y = Y_i \frac{a}{|a|}\left(\left(\frac{y}{h}+\frac{1}{2}\right)^p - \left(\frac{-y}{h}+\frac{1}{2}\right)^p\right) \quad (2)$$

Which E_i and Y_i are the magnitude of E , Y at top and bottom of the beam, h is the beam height and n, p are the material parameters. Figure 3 shows the variation of the young's modules along y direction. Transvers loadingisdefined as follow;

$$q(x) = ax^m \quad (3)$$

The normal stress is defined by the Hook's law;

$$\sigma_x = E\varepsilon_x \quad (4)$$

$$\varepsilon_x = \frac{y}{R} \quad (5)$$

$$\frac{1}{R} = \frac{d^2\omega}{dx^2} \quad (6)$$

Which in above relation R is curvature radius and is the axial strain. So, we can rewrite the normal stress as follow;

$$\sigma_x = Ey\frac{d^2\omega}{dx^2} \quad (7)$$

Strain energy density regarding to Euler-Bernoulli beam by neglecting the shear strain energy density can be express;

$$U = \frac{\sigma_x^{\ 2}}{2E} = \frac{E}{2}\left(\frac{d^2\omega}{dx^2}\right)y^2 \quad (8)$$

The total strain energy stored in an elastic solid occupying a region V is then given by the integral over the domain:

$$U_T = \int_0^L \left[\iint_A \frac{E}{2}\left(\frac{d^2\omega}{dx^2}\right)y^2 dA\right]dx \quad (9)$$

$$k = \iint_A Ey^2 dA \quad (10)$$

$$k = \frac{E_i bh^3}{2(n+1)(n+2)(n+3)}\left[n^2 + n + 2 + \left(\frac{1}{2}\right)^n\right] \quad (11)$$

$$U_T = \int_0^L \left[\frac{k}{2}\left(\frac{d^2\omega}{dx^2}\right)^2\right]dx \quad (12)$$

The work done by external forces is obtaine as follow;

$$W = \int_0^L q\omega dx - \left[V_0\omega - M_0\frac{d\omega}{dx}\right]_0^L \quad (13)$$

Therefore, according to Figure 1, the total potential energy for this beam case is given by;

$$\Pi = U_T - W = \int_0^L \left[\frac{k}{2}\left(\frac{d^2\omega}{dx^2}\right)^2\right]dx - \int_0^L q\omega dx + \left[V_0\omega - M_0\frac{d\omega}{dx}\right]_0^L \quad (14)$$

According to calculus of variation, the first variation of this quantity must vanish;

$$\delta\Pi = 0 \quad (15)$$

$$\delta\Pi = \frac{k}{2}\int_0^L 2\frac{d^2\omega}{dx^2}\frac{d^2\delta\omega}{dx^2}dx - \int_0^L q\delta\omega dx - \left[V_0\omega - M_0\frac{d\omega}{dx}\right]_0^L \quad (16)$$

Now the first integral term can be integrated by parts twice to get

$$\delta\Pi = \int_0^L \left(k\left(\frac{d^4\omega}{dx^4}\right) - q\right)\delta\omega dx + \left[\frac{d\delta\omega}{dx}(M - M_0) - \delta\omega(V - V_0)\right]_0^L = 0 \quad (17)$$

The integral and boundary terms must all vanish, thus implying;

$$\int_0^L \left(k\left(\frac{d^4\omega}{dx^4}\right) - q \right)\delta\omega = 0 \quad (18)$$

$$\begin{cases} x = 0, l \quad \delta\omega = 0 \quad or \quad V = V_0 \\ x = 0, l \quad \delta\left(\frac{d\omega}{dx}\right) = 0 \quad or \quad M = M_0 \end{cases} \quad (19)$$

For this integral to vanish for all variations $\delta\omega$, the fundamental lemma in the calculusof variations implies that integrand must be zero, giving;

$$k\left(\frac{d^4\omega}{dx^4}\right) - q = 0 \quad (20)$$

The deflection of the beam will be determined by solving the above equation;

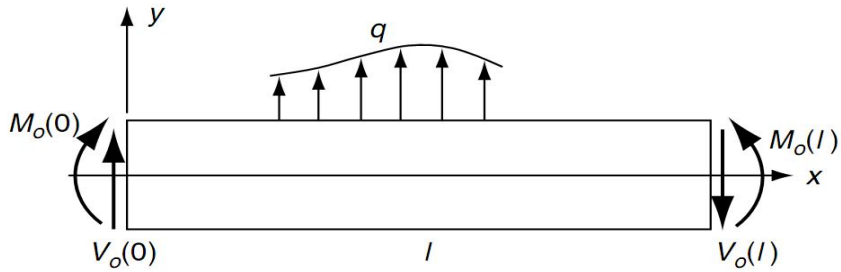

Fig .1. Euler-Bernoulli beam geometry and external forces.

$$\omega = \begin{bmatrix} \frac{a}{k(m+1)(m+2)(m+3)(m+4)} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} . \begin{bmatrix} x^{m+4} & \frac{x^3}{6} & \frac{x^2}{2} & x & 1 \end{bmatrix} \quad (21)$$

By considering the boundry condition, the constants will be obtain;

$$\begin{cases} x = 0 \quad \omega = 0 \quad and \quad M = k\frac{d^2\omega}{dx^2} = 0 \\ x = L \quad \omega = 0 \quad and \quad M = k\frac{d^2\omega}{dx^2} = 0 \end{cases} \quad (22)$$

$$[C] = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} -\frac{aL^{m+1}}{k(m+1)(m+2)} \\ 0 \\ \frac{aL^{m+3}(m+6)}{6k(m+2)(m+3)(m+4)} \\ 0 \end{bmatrix} \quad (23)$$

And the stress will be determined as follow;

$$\sigma_x = \frac{E_i a\left(\left|\frac{y}{h}\right| + \frac{1}{2}\right)^n y}{k(m+1)(m+2)}\left(L^{m+1}x - x^{m+2}\right) \quad (24)$$

The required moment to produce this stress in beam in Elastic region obtained;

$$M_E = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x y b dy = \frac{d^2\omega}{dx^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} E y^2 b dy = k \frac{d^2\omega}{dx^2} \quad (25)$$

We assume that up to y' in the beam is elastic and the rest is plastic (Figure 2), the moment for this elastoplastic for this deformation is;

$$M_{EP} = 2\int_{y'}^{\frac{h}{2}} Y y b dy + \int_{-y'}^{y'} \sigma_x y b dy \quad (26)$$

$$M_{EP} = 2\int_{y'}^{\frac{h}{2}} Y_i b \left(\left(\frac{y}{h} + \frac{1}{2} \right)^p + \left(-\frac{y}{h} + \frac{1}{2} \right)^p \right) y dy + 2\int_0^{y'} E_i b \left(\left| \frac{y}{h} \right| + \frac{1}{2} \right)^n y^2 \frac{d^2\omega}{dx^2} dy \quad (27)$$

$$M_{EP} = 2Y_i b \left[\frac{ph^2}{2(p+1)(p+2)} - \frac{y'h\left(\left(\frac{y}{h} + \frac{1}{2} \right)^{p+1} + \left(-\frac{y}{h} + \frac{1}{2} \right)^{p+1} \right)}{p+1} + \frac{h^2 \left(\left(\frac{y}{h} + \frac{1}{2} \right)^{p+2} - \left(-\frac{y}{h} + \frac{1}{2} \right)^{p+2} \right)}{(p+1)(p+2)} \right] \quad (28)$$

$$+2E_i ab \left[\frac{y'^2 h \left(\frac{y'}{h} + \frac{1}{2} \right)^{n+1}}{n+1} - \frac{2y'h^2 \left(\frac{y'}{h} + \frac{1}{2} \right)^{n+2}}{(n+1)(n+2)} + \frac{2h^3 \left(\frac{y'}{h} + \frac{1}{2} \right)^{n+3}}{(n+1)(n+2)(n+3)} - \frac{2h^3 \left(\frac{1}{2} \right)^{n+3}}{(n+1)(n+2)(n+3)} \right]$$

The moment require for a section to become fully plastic is;

$$M_P = Y_i b \left[\frac{ph^2}{(p+1)(p+2)} \right] \quad (29)$$

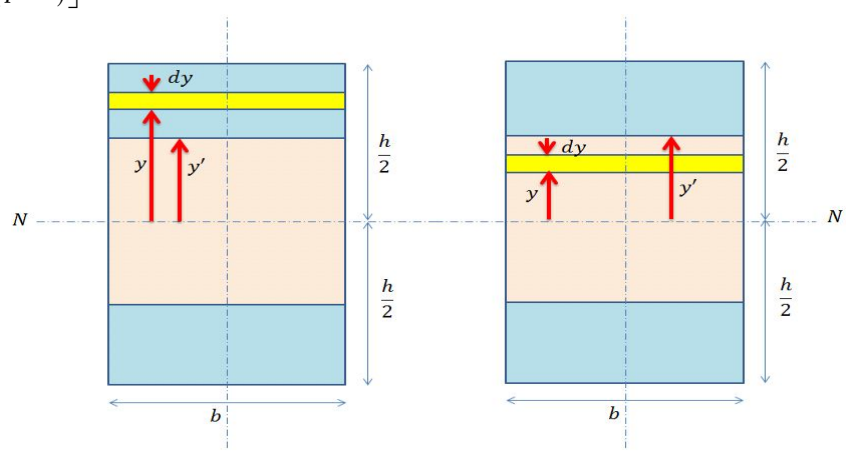

Fig.2. Elastic and plastic region of cross section of the beam

3. RESULTS AND DISCUSSION

For a simply support beam subjected to transverse loading $q = ax^2$ $(m = 2)$, height $h = 0.4m$, thickness $b = 0.2m$,length $L = 2m$ and with material property $n = p = 3$, $E_i = 70Gpa$, $Y_i = 95MPa$ the magnitude of critical a to start yielding from top and bottom at both ends of the beam is $-6.1658 \times 10^5 \frac{N}{M}$. As a increases, the yield region increases from top and bottom to natural axis in y direction .along x direction, the plastic region starts from the mid length and expands to other ends. Figure 5 shows stress in 3D space for this case and figure 6 illustrates stress changes in x direction respect to y at $x = \frac{l}{2}$, intersecting σ_x and Y shows that yielding starts at $y = \pm\frac{h}{2}$. The require moment to start yielding is equal 4.1800×10^5. Figure 4 and 7illustrate the deflection and slope of the beam. For the above considering beam. Maximum displacement is equal $\delta_{\max} = -2.9055 \times 10^{-3} m$ which at $x = 1.071m$ is occurred. This is shown in figure 4. In this point the slop of the beam is zero. As loading is increase,$(a = -8 \times 10^5)$, the beam remains elastic up to y′= ± 0.153, and from y′ up to top of the beam is plastic.Figure 7, shows this state.

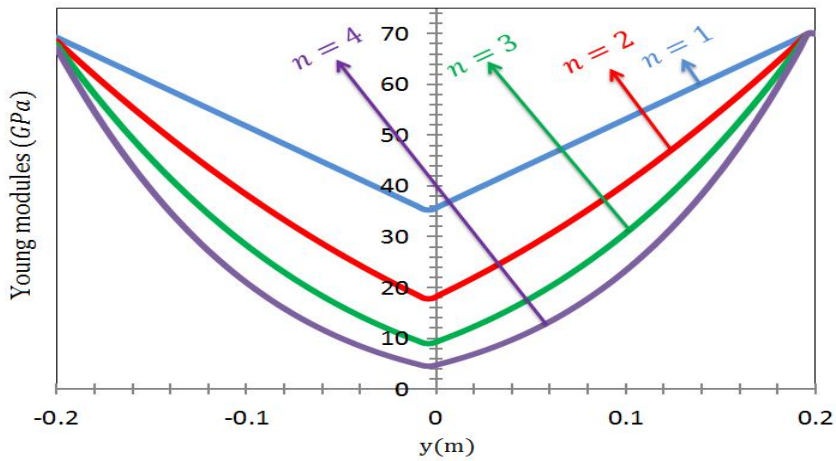

Fig.3. Distribution of Young's modulus along Y direction

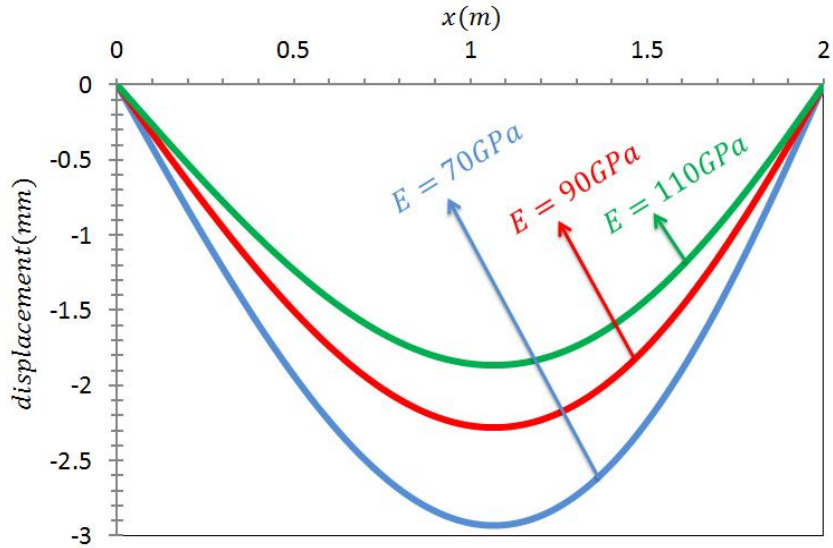

Fig.4. Distribution of deflection along the beam

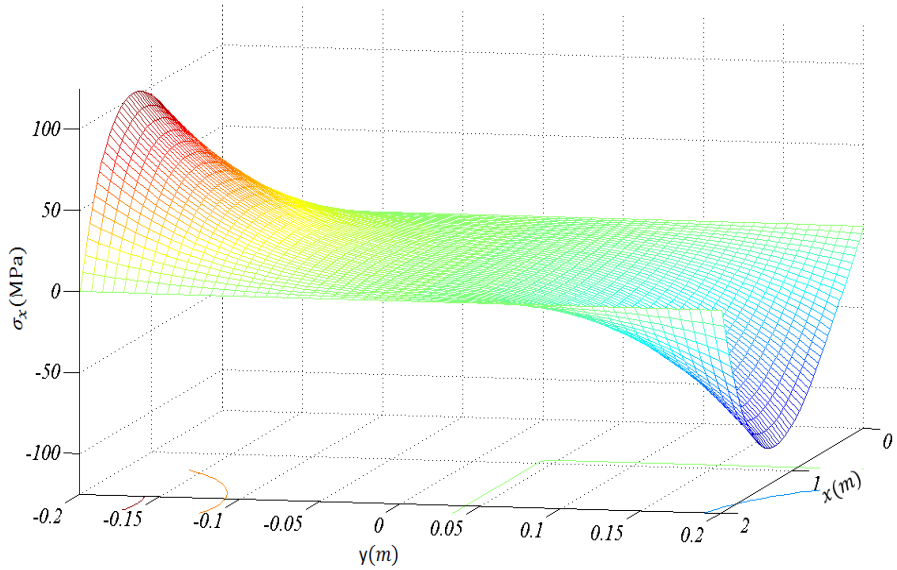

Fig.5. Distribution of axial stress of FGM beam (3D)

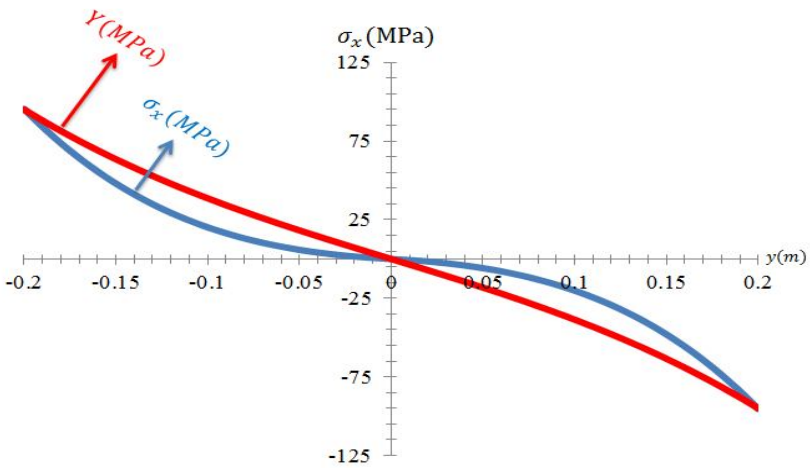

Fig.6. Comparison of axial stress versus yield stressof the beam along x axes at $x = \frac{L}{2}$

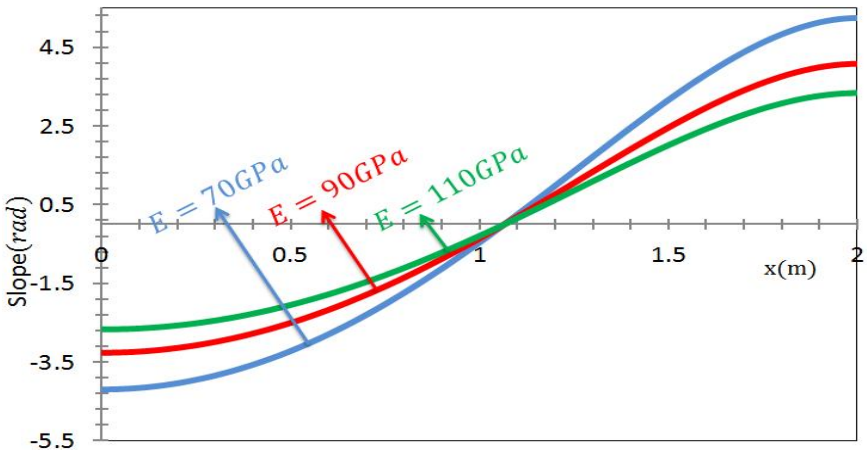

Fig.7. Distribution of slop along the beam

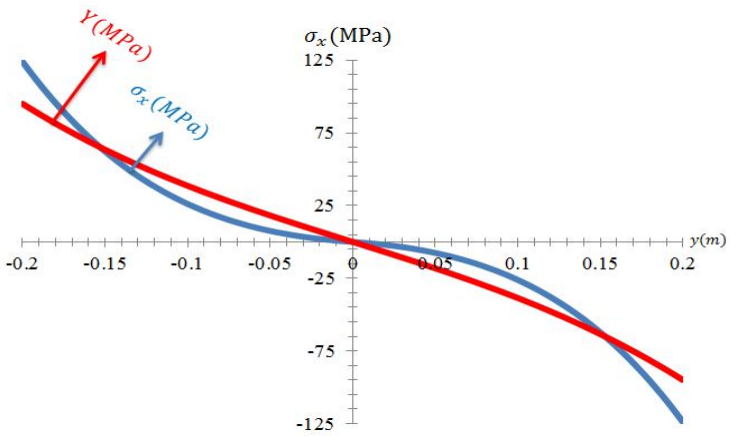

Fig.8. Illustration of plastic and elastic region

4. Conclusion

This paper shows that, for an elastoplastic FGM Euler-Bernoulli beam subjected to a transverse loading, when the loading starts, the beam is fully elastic,as loading increases, the beam starts becoming plastic from top and end and it continues till the beam become fully plastic. The plastic region along x axes starts from middle of the beam

length, and expand to other two ends. Also, deflection, stress, and maximum bending to have fully plastic beam has been determined. These parameters will be useful for designing of a FGM beam.

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