J. Basic. Appl. Sci. Res., 1(11)2435-2441, 2011

ISSN 2090-4304
Journal of Basic and Applied Scientific Research
www.textroad.com

An Elasticity Solution for Functionally Graded Hollow Disks under Radially Symmetry Loads

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ABSTRACT

An accurate and efficient solution procedure based on the elasticity theory is employed to investigate the elastic behavior of disks made of functionally graded materials subjected to mechanical loads. The material properties are assumed to be graded in the thickness direction as a parabolic function. In this study, the disk is subjected to constant uniform pressure and the Poisson's ratio, is assumed as constant. Using the finite element method (FEM), numerical solution is obtained.

KEY WORDS: Disk, Functionally Graded Material (FGM), Finite Element Method (FEM), Parabolic.

1. INTRODUCTION

Functionally graded materials (FGMs) are composite materials with the mechanical properties changing continuously from one surface to the other.

A paper was also published by Horgan and Chan [1] where it was noted that increasing the positive exponent of the radial coordinate provided a stress shielding effect whereas decreasing it created stress amplification. Closed-form solutions are obtained by Tutuncu and Ozturk [2] for cylindrical and spherical vessels with variable elastic properties obeying a simple power law through the wall thickness which resulted in simple Euler-Cauchy equations whose solutions were readily available. Based on the assumption that Poisson's ratio is constant and modulus of elasticity is an exponential function of radius, Chen and Lin [3] have analyzed stresses and displacements in FG cylindrical and spherical pressure vessels. Zamani Nejad et al. [4] developed 3-D set of field equations of FGM thick shells of revolution in curvilinear coordinate system by tensor calculus. Using plane elasticity theory and complementary functions method, Tutuncu and Temel [5] are obtained axisymmetric displacements and stresses in functionally-graded hollow cylinders, disks and spheres subjected to uniform internal pressure. An analytical solution is developed to determine deformations and stresses in circular disks made of functionally graded materials subjected to internal and/or external pressure [6]. Ghannad and Zamani Nejad [7] obtained the elastic solution of clamped-clamped thick-walled cylindrical shells by an analytic method. Assuming the volume fractions of two phases of a FG material (FGM) vary only with the radius, Nie et al. [8] obtained a technique to tailor materials for functionally graded (FG) linear elastic hollow cylinders and spheres to attain through-the-thickness either a constant circumferential (or hoop) stress or a constant in-plane shear stress.

In this study, a two-dimensional elasticity solution and a numerical solution using finite element method for elastic analysis of FGM parabolic disk are presented.

2. FORMULATION OF PROBLEM

Consider a FGM disk with an inner radius a, and an outer radius b, subjected to internal and external pressure P_i and P_o, respectively. (Fig. 1).

It is assumed that the Poison's ratio υ, takes a constant value and the modulus of elasticity E, is assumed to vary radially according to parabolic form as follows [9],

$$E(R) = E_i\left(1 + n\frac{1-R^{\eta}}{1-K^{\eta}}\right) \quad , \quad R = \frac{r}{a} \quad , \quad n = \frac{E_o}{E_i} - 1 \quad , \quad K = \frac{b}{a} \tag{1}$$

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where E_i and E_o are modulus of elasticity in inner and outer surfaces, respectively. Here, n and η are material parameters.

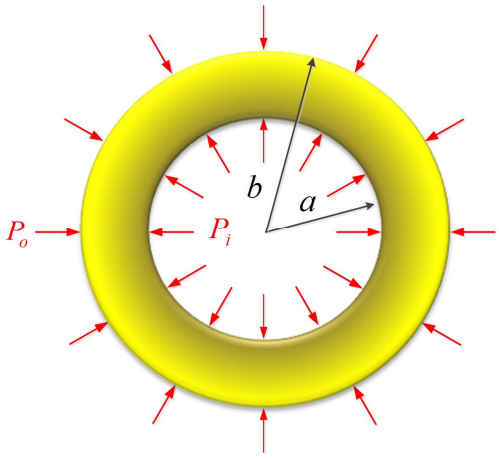

Fig. 1. Configuration of a FGM disk with radially varying properties.

Radial and circumferential strains $(\varepsilon_r, \varepsilon_\theta)$, in the polar coordinates are as follows,

$$\varepsilon_r = \frac{du}{dr} \tag{2}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{3}$$

where u is radial displacement.

The stress-strain relations for non-homogenous and isotropic materials are

$$\begin{pmatrix} \sigma_r \\ \sigma_\theta \end{pmatrix} = \frac{E(R)}{a} \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} \frac{du}{dR} \\ \frac{u}{R} \end{pmatrix} \tag{4}$$

where σ_r and σ_θ are radial and circumferential stresses. A and B are related to Poisson's ratio υ as

$$\begin{cases} A = \dfrac{1}{1-\upsilon^2} \\ B = \dfrac{\upsilon}{1-\upsilon^2} \end{cases} \tag{5}$$

The equilibrium equation in the absence of body forces, is expressed as

$$\frac{d\sigma_r}{dR} + \frac{\sigma_r - \sigma_\theta}{R} = 0 \tag{6}$$

here, prime denotes differentiation with respect to R.

Substituting Eqs. (4), into Eq. (6), the equilibrium equation is expressed as

$$\begin{cases} R^2 \dfrac{d^2u}{dR^2} + R\left(1 + \dfrac{RE'}{E}\right)\dfrac{du}{dR} - \left(1 - \upsilon^* \dfrac{RE'}{E}\right)u = 0 \\ \upsilon^* = \dfrac{B}{A} \end{cases} \tag{7}$$

The general solution of Eq. (7) is as follows

$$u(R) = C_1 G(R) + C_2 H(R) \tag{8}$$

where C_1 and C_2 are arbitrary integration constants. Here G and H are homogeneous solutions.

Substituting Eq. (8) into Eqs. (4), yields

$$\begin{pmatrix}\sigma_r\\ \sigma_\theta\end{pmatrix}=\frac{E(R)}{a}\begin{pmatrix}A & B\\ B & A\end{pmatrix}\begin{pmatrix}C_1G'+C_2H'\\ C_1\dfrac{G}{R}+C_2\dfrac{H}{R}\end{pmatrix} \tag{9}$$

The forms of G and H will be determined next.

Substituting Eq. (1) into Eq. (7), the governing differential equation is as follows

$$R^2\left(1-\frac{nR^\eta}{n+K^\eta-1}\right)\frac{d^2u}{dR^2}+R\left[1-\frac{(1+\eta)nR^\eta}{n+K^\eta-1}\right]\frac{du}{dR}-\left[1-\frac{(1-\upsilon^*\eta)nR^\eta}{n+K^\eta-1}\right]u=0 \tag{10}$$

Equation 10 is a homogeneous hypergeometric differential equation.

Using a new variable $x=mR^\eta=\left(n/(n+K^\eta-1)\right)R^\eta$ and applying the transformation $u(R)=Ry(x)$, the result Eq. (10) is

$$x(1-x)\frac{d^2y}{dx^2}+\left[\left(1+\frac{2}{\eta}\right)-2\left(1+\frac{1}{\eta}\right)x\right]\frac{dy}{dx}-\frac{1+\upsilon^*}{\eta}y=0 \tag{11}$$

The solution of Eq. (11) is given as

$$y(x)=C_1F_H(\alpha,\beta,\delta;x)+\bar{C}_2x^{-2/\eta}F_H(\alpha-\delta+1,\beta-\delta+1,2-\delta;x) \tag{12}$$

With $F_H(\alpha,\beta,\delta;x)$ being the hypergeometric function defined by Abramowitz and Stegun [10],

$$F_H(\alpha,\beta,\delta;x)=1+\sum_{k=1}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\delta)_k}\frac{x^k}{k!} \tag{13}$$

where

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) \tag{14}$$

Thus

$$F_H(\alpha,\beta,\delta;x)=1+\frac{\alpha\beta}{\delta}\frac{x}{1!}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\delta(\delta+1)}\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\delta(\delta+1)(\delta+2)}\frac{x^3}{3!}+\cdots \tag{15}$$

In Eq. (15), the arguments α, β, and δ are determined as

$$\begin{cases}\alpha=\dfrac{2(1+\upsilon^*)}{(2+\eta)+\sqrt{(2+\eta)^2-4\eta(1+\upsilon^*)}}\\[2ex] \beta=\dfrac{2(1+\upsilon^*)}{(2+\eta)-\sqrt{(2+\eta)^2-4\eta(1+\upsilon^*)}}\\[2ex] \delta=1+\dfrac{2}{\eta}\end{cases} \tag{16}$$

From $u(R)=Ry(mR^\eta)$, the homogeneous solutions G and H are found in the form

$$G(R)=RF_H(\alpha,\beta,\delta;mR^\eta) \tag{17}$$

$$H(R)=\frac{1}{R}F_H(\alpha-\delta+1,\beta-\delta+1,2-\delta;mR^\eta) \tag{18}$$

The Eqs. (8) and (9) may be rewritten with non-dimensional parameters as

$$U(R)=C_3G(R)+C_4H(R) \tag{19}$$

$$\begin{pmatrix}\bar{\sigma}_r\\ \bar{\sigma}_\theta\end{pmatrix}=\left(1+n\frac{1-R^\eta}{1-K^\eta}\right)\begin{pmatrix}A & B\\ B & A\end{pmatrix}\begin{pmatrix}C_3G'+C_4H'\\ C_3\dfrac{G}{R}+C_4\dfrac{H}{R}\end{pmatrix} \tag{20}$$

where

$$U=\frac{uE_i}{aP_i}\quad,\quad \bar{\sigma}=\frac{\sigma}{P_i}\quad,\quad \frac{C_3}{C_1}=\frac{C_4}{C_1}=\frac{E_i}{aP_i} \tag{21}$$

Integration constants C_3 and C_4 are determined by using the following boundary conditions

$$\bar{\sigma}_r(R=1)=-1 \quad , \quad \bar{\sigma}_r(R=K)=-\frac{P_o}{P_i}=-P \tag{22}$$

Thus

$$\begin{cases} \left[AG'(1)+BG(1)\right]C_3+\left[AH'(1)+BH(1)\right]C_4=-1 \\ \left[AG'(K)+B\dfrac{G(K)}{K}\right]C_3+\left[AH'(K)+B\dfrac{H(K)}{K}\right]C_4=-\dfrac{P}{1+n} \end{cases} \tag{23}$$

Using Eqs. (23), the constants C_3 and C_4 are determined as follows

$$C_3=\frac{D_2D_5+D_4}{D_2D_3-D_1D_4} \quad , \quad C_4=\frac{D_1D_5+D_3}{D_1D_4-D_2D_3} \tag{24}$$

where

$$\begin{cases} D_1=AG'(1)+BG(1) \\ D_2=AH'(1)+BH(1) \\ D_3=AG'(K)+B\dfrac{G(K)}{K} \\ D_4=AH'(K)+B\dfrac{H(K)}{K} \\ D_5=-\dfrac{P}{1+n} \end{cases} \tag{25}$$

3. RESULTS AND DISCUSSION

In this part, for the purpose of illustration, a case study is investigated. For example, assume that $b=1.5a$, $P_i=P_o$ and $\upsilon=0.3$.

In this paper, for finite element modeling of FGM disk, an innovative application for multilayering of wall thickness in the radial direction has been performed. Homogenous layers which are of identical thickness and step-variable elasticity modulus have been formed by this method.

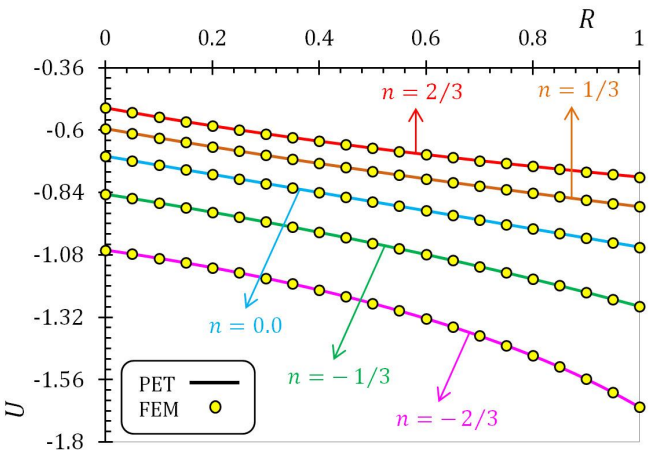

Fig. 2. Radial distribution of radial displacement ($\eta=1.2$).

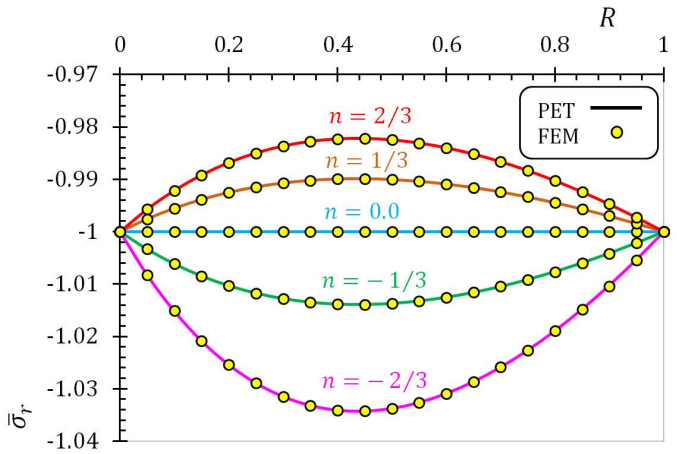

Fig. 3. Radial distribution of radial stress ($\eta = 1.2$).

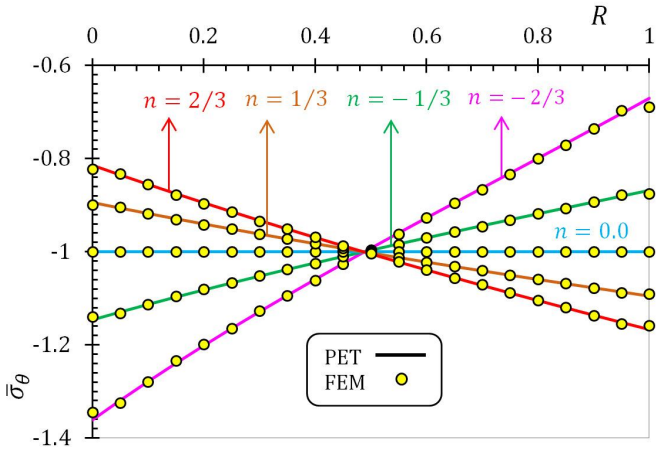

Fig. 4. Radial distribution of circumferential stress ($\eta = 1.2$).

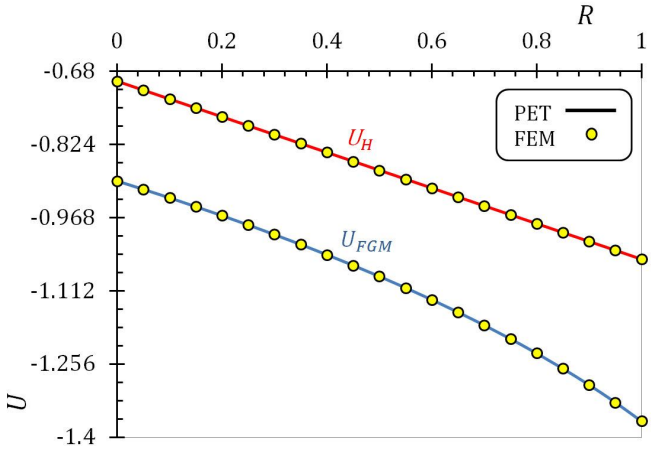

Fig. 5. Comparison of radial displacement in a FGM disk ($n = -0.5$, $\eta = 3.6$) to those in homogeneous disk ($n = 0$).

For different values of material inhomogeneity parameter n , radial displacement, radial stress, and circumferential stress along the radial direction are plotted in Figures 2-5.

In Figs. 2 and 3, distribution of the radial displacement and the radial stress along the radial direction for different values of n and $\eta = 1.2$ are shown. It is seen from the curves that at the same position $(0 < R < 1)$, for higher values of n, radial displacement and radial stress decrease.

The circumferential stress along the radial direction for different values of n and $\eta = 1.2$ is plotted in Fig. 4. It must be noted from this figure that at the same position, almost for $R < 0.5$, there is an decrease in the value of the circumferential stress as n increases, whereas for $R > 0.5$ this situation was reversed. Besides, along the radial direction for the positive magnitudes of n the circumferential stress increases, while for negative magnitude of n, the circumferential stress decreases.

In Figs. 5 and 6, the stresses and radial displacement using values $n = -0.5$ and $\eta = 3.6$, is calculated and compared to those in a homogeneous disk ($n = 0$).

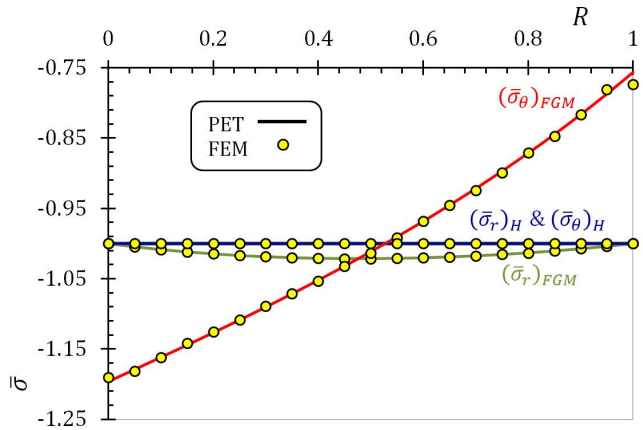

Fig. 6. Comparison of stresses in a FGM disk ($n = -0.5$, $\eta = 3.6$) to those in homogeneous disk ($n = 0$).

4. CONCLUSION

In the field of the infinitesimal theory of elasticity, an axisymmetric pressurized parabolic FGM disk is investigated. To show the effect of inhomogeneity on the stress distributions, different values were considered for material inhomogeneity parameter n. The presented results show that the material inhomogeneity has a significant influence on the mechanical behaviors of disks made of parabolic FG. In this study, in order to numerical analysis of problem, an axisymmetric element has been applied for modeling and meshing. Good agreement was found between the analytical solutions and the solutions carried out through the FEM.

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