J. Appl. Environ. Biol. Sci., 8(7)26-35, 2018
© 2018, TextRoad Publication

ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

A New Third Order Iterative Integrator for Cauchy Problems

Muhammad Daud Kandhro[1], Asif Ali Shaikh[2], Sania Qureshi[3]

Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan

Received: January 14, 2018
Accepted: April 1, 2018

ABSTRACT

In this paper, an explicit and single step Runge-Kutta iterative integrator of third order has been developed which is used to solve both autonomous and non-autonomous type of initial value problems also called Cauchy problems in ordinary differential equations. Linear stability analysis with corresponding stability region is drawn and error analysis has been provided to confirm third order accuracy of the integrator. Inclusion of a partial derivative with respect to the dependent variable within two slopes of the integrator has improved its efficiency in terms of local and global truncation errors. Finally, numerical examples are provided to show performance of the proposed integrator in comparison with other existing methods having same order of local accuracy. The software MATLAB R2017b was employed in order to produce all the numerical results and graphical illustrations presented in this paper whereas the MATLAB code designed to get such numerical results for all the integrators under consideration have also been provided.

KEYWORDS: Runge-Kutta, Stability, Local Truncation Error, Non-autonomous, Cauchy problems.

1. INTRODUCTION

Ordinary differential equations usually express all natural phenomena we come across in this physical universe. Their usage is present everywhere in biology, environmental engineering, physical systems, business, and economics to name a few [1-3]. It is a common practice of physical and biological researchers around the world to study the mathematical modeling of various physical problems based upon Cauchy problems such as radioactive decay, population dynamics, mechanical systems, fluid flows, electrical networks, rate of chemical reactions and many more [4-6]. Many problems of mathematical physics can be represented in the form of such ordinary differential equations.

In many situations, the methods capable to obtain solution of certain mathematical models stand to be very hard and complicated and sometimes even fail to produce the required results [7-9]. There are various numerical methods to get approximate solution of an initial value problem in ordinary differential equations and this is because only one numerical method cannot serve the purpose to get the solution of every type of initial value problem. Specially, numerous nonlinear types of Cauchy problems have stimulated researchers to get either the new numerical methods or improve the existing ones [10-17]. Another reason is the computational effort and CPU time required by the methods to solve the problem. The authors in [18] have tried to reduce the number of slope evaluations per integration step for autonomous initial value problems whereas the authors in [19] have extended the research work to non-autonomous type of problems. Moreover, nonlinear iterative integrators are suitable for the initial value problems having singular solutions along the integration interval under consideration as discussed in [20-24].

We consider the general first order ordinary differential equation with an initial condition, also called Cauchy Problem, as given below:

$$\frac{dy}{dx} = f\left(x, y\right), \; y\left(x_0\right) = y_0 \tag{1}$$

Existence of unique solution of (1) is assumed for the integration interval of $x \in \left[x_0, x_n\right]$. Here, exact solution is denoted by $y\left(x_n\right)$ whereas the numerical solution is by y_n taking the step size $h = \frac{x_n - x_0}{N}$, where $N = 1, 2, 3, \ldots$

2. DERIVATION OF THE PROPOSED INTEGRATOR

The general form of a single-step explicit numerical integrator to solve an initial value problem is given as:

$$y_{n+1} = y_n + h\phi_f\left(x_n, y_n; h\right) \tag{2}$$

Corresponding Author: Sania Qureshi, Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan. **Email:** sania.qureshi@faculty.muet.edu.pk

where $\phi_f(x_n, y_n; h)$ can be expressed in terms of Taylor series expansion of an arbitrary function $f(x, y)$ as follows:

$$\phi_f(x_n, y_n; h) = \sum_{p=0}^{\infty} \frac{h^p}{(p+1)!}\left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)^p f(x, y) \tag{3}$$

Further, the Taylor series expansion of $y(x_n + h)$ is of the form

$$\begin{aligned} y(x_n + h) &= y(x_n) + hf + \frac{1}{2!}h^2\left(f_x + ff_y\right) + \frac{1}{3!}h^3\left(f_{xx} + 2ff_{xy} + f^2 f_{yy} + ff_y^2 + f_x f_y\right) \\ &+ \frac{1}{4!}h^4\begin{pmatrix} f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + 5ff_y f_{xy} + 3f_x f_{xy} + f^3 f_{yyy} \\ + 4f^2 f_y f_{yy} + 3ff_x f_{yy} + ff_y^3 + f_x f_y^2 + f_{xx} f_y \end{pmatrix} + O\left(h^5\right) \end{aligned} \tag{4}$$

The proposed integrator of the present article is of the form:

$$y_{n+1} = y_n + h\phi_{3stageRK}(x_n, y_n; h) \tag{5}$$

where

$$\begin{gathered} \phi_{3stageRK}(x_n, y_n; h) = b_1 k_1 + b_2 k_2 + b_3 k_3 \\ k_1 = f(x_n, y_n),\ k_2 = f\left(x_n + a_2 h, y_n + hk_1\left(b_{21} + hc_{21} f_y\right)\right) \\ k_3 = f\left(x_n + a_3 h, y_n + h\left(b_{31}k_1 + b_{32}k_2\right) + h^2 c_{31} k_1 f_y\right) \end{gathered}$$

Expanding k_2 and k_3 in Taylor's series, we obtain

$$\begin{aligned} k_2 &= f + \left(ff_y b_{2,1} + f_x a_2\right)h + \left(\frac{1}{2}f_{y,y}f^2 b_{2,1}^2 + f_{x,y} f a_2 b_{2,1} + ff_y^2 c_{2,1} + \frac{1}{2}f_{x,x}a_2^2\right)h^2 + \\ &\left(\frac{1}{6}f_{y,y,y}f^3 b_{2,1}^3 + \frac{1}{2}f_{x,y,y}f^2 a_2 b_{2,1}^2 + f_{y,y}f^2 f_y b_{2,1}c_{2,1} + \frac{1}{2}f_{x,x,y} f a_2^2 b_{2,1} + f_{x,y}ff_y a_2 c_{2,1} + \frac{1}{6}f_{x,x,x}a_2^3\right)h^3 + \mathrm{O}\left(h^4\right) \end{aligned}$$

$$\begin{aligned} k_3 &= f + \left(f_y f b_{32} + f_y f b_{31} + f_x a_3\right)h + \begin{pmatrix} \frac{1}{2}f_{yy}f^2 b_{32}^2 + f_{yy}f^2 b_{31}b_{32} + \frac{1}{2}f_{yy}f^2 b_{31}^2 + f_{xy} f a_3 b_{32} + \\ f_{xy} f a_3 b_{31} + f_y^2 f c_{31} + \frac{1}{2}f_{xx}a_3^2 + f_y^2 f b_{21}b_{32} + f_x f_y a_2 b_{32} \end{pmatrix}h^2 \\ &+ \begin{pmatrix} \frac{1}{2}f_{yyy}f^3 b_{31}b_{32}^2 + \frac{1}{2}f_{yyy}f^3 b_{31}^2 b_{32} + \frac{1}{2}f_{xyy}f^2 a_3 b_{32}^2 + \frac{1}{2}f_{xyy}f^2 a_3 b_{31}^2 + \frac{1}{2}f_{xxy} f a_3^2 b_{32} + \frac{1}{2}f_{xxy} f a_3^2 b_{31} + \\ \frac{1}{2}f_y f^2 f_{yy} b_{21}^2 b_{32} + f_y^3 f b_{32} c_{21} + f_y ff_{xy} a_2 b_{21} b_{32} + \frac{1}{2}f_y f_{xx} a_2^2 b_{32} + \frac{1}{6}f_{yyy}f^3 b_{32}^3 + \frac{1}{6}f_{yyy}f^3 b_{31}^3 + \\ f_{xyy}f^2 a_3 b_{31} b_{32} + f_{yy}f^2 f_y b_{31} c_{31} + f_{xy}ff_y a_3 c_{31} + f_y f^2 f_{yy} b_{21} b_{31} b_{32} + f_x ff_{yy} a_2 b_{31} b_{32} + \frac{1}{6}f_{xxx}a_3^3 + \\ f_y f^2 f_{yy} b_{21} b_{32}^2 + f_x ff_{yy} a_2 b_{32}^2 + f_y ff_{xy} a_3 b_{21} b_{32} + f_x f_{xy} a_2 a_3 b_{32} + f_{yy} f^2 f_y b_{32} c_{31} \end{pmatrix}h^3 + O\left(h^4\right) \end{aligned}$$

Substituting the result of k_1 , k_2 and k_3 into (5) then equate the coefficients of powers of *h* up to h^3 with that of (4) to obtain the following order conditions:

$$
\begin{aligned}
&b_1+b_2+b_3=1 && a_2b_3b_{32}=\frac{1}{6} && a_2b_2+a_3b_3=\frac{1}{2}\\
&\frac{1}{2}\left(a_2^2b_2+a_3^2b_3\right)=\frac{1}{6} && b_2b_{21}+b_3b_{31}+b_3b_{32}=\frac{1}{2} && a_2b_2b_{21}+a_3b_3b_{31}+a_3b_3b_{32}=\frac{1}{3}\\
&b_2c_{21}+b_3c_{31}+b_3b_{21}b_{32}=\frac{1}{6} && \frac{1}{2}\left(b_2b_{21}^2+b_3b_{31}^2+b_3b_{32}^2\right)+b_3b_{31}b_{32}=\frac{1}{6}
\end{aligned}
\tag{6}
$$

One of the solutions of the above nonlinear system (6) forms the proposed three-stage explicit RK iterative integrator of third order as given below:

$$
\begin{aligned}
k_1&=f\left(x_n,y_n\right)\\
k_2&=f\left(x_n+\frac{2}{3}h,y_n+\frac{2}{3}hk_1+h^2k_1f_y\right)\\
k_3&=f\left(x_n+\frac{2}{3}h,y_n-\frac{1}{3}hk_1+hk_2-2h^2k_1f_y\right)\\
y_{n+1}&=y_n+\frac{1}{4}h\left(k_1+2k_2+k_3\right)
\end{aligned}
\tag{7}
$$

The above proposed iterative integrator (7) can be used to solve both autonomous and non-autonomous type of initial value problems in ordinary differential equations. After getting this new integrator, we will analyze it for its accuracy, convergence, order of consistency and linear stability. These are the important terms related to an iterative integrator for it to be acceptable in the field of computational and applied mathematics as proved in [25].

3. ERROR ANALYSIS

In order to obtain the local truncation error of the proposed integrator, a usual functional associated to the integrator has been considered, that is given below:

$$L(z(x),h)=z(x+h)-y_{n+1}$$

where $z(x)$ is an arbitrary function defined along the integration interval $[x_0,x_n]$ and differentiable as many times as required. Having expanded it into Taylor series about x and collecting the terms in h, the local truncation error under local assumption of the following form has been obtained that ensures at least third order accuracy of the proposed integrator:

$$
T_{n+1}=\begin{pmatrix}
-\frac{5}{24}f_y^3f-\frac{1}{72}f_{x,x}f_y+\frac{1}{24}f_xf_y^2+\frac{1}{72}f_{x,x,y}f+\frac{1}{216}f_{y,y,y}f^3+\\
\frac{1}{72}f_{x,y,y}f^2+\frac{1}{72}f_{x,y}f_x+\frac{1}{216}f_{x,x,x}-\frac{1}{72}f_{x,y}f_yf+\frac{1}{72}f_{y,y}f_xf
\end{pmatrix}h^4+O\left(h^5\right)
\tag{8}
$$

4. CONSISTENCY ANALYSIS

Definition 4.1 Given an initial value problem $y'(x)=f\left(x_n,y_n\right);\ y\left(x_0\right)=y_0$; an iterative integrator with an increment function $\Phi\left(x_n,y_n;h\right)$ is said to be consistent if

$$\lim_{h\to 0}\Phi\left(x_n,y_n;h\right)=f\left(x_n,y_n\right)$$

The increment function of the proposed integrator (7) is shown as:

$$
\begin{aligned}
\lim_{h\to 0}\Phi\left(x_n,y_n;h\right) &= \frac{1}{4}\lim_{h\to 0}\left(k_1+2k_2+k_3\right)\\
&= \frac{1}{4}\lim_{h\to 0}\left[\begin{array}{l} f\left(x_n,y_n\right)+2f\left(x_n+\frac{2}{3}h,y_n+\frac{2}{3}hk_1+h^2k_1f_y\right)+\\ f\left(x_n+\frac{2}{3}h,y_n-\frac{1}{3}hk_1+hk_2-2h^2k_1f_y\right)\end{array}\right]\\
&= f\left(x_n,y_n\right)
\end{aligned}
$$

Thus, the proposed integrator is shown to be **consistent** with at least **third order accuracy**.

5. LINEAR STABILITY ANALYSIS

An iterative integrator should not produce entirely different results for very small changes in the input data, that is, it should be stable in order to be acceptable for use in solving practical problems in computational and applied mathematics. Numerical stability of an iterative integrator ensures the control of the magnitude of errors inherent to either the integrator or the initial value problem under consideration. Among various ways to check stability of the iterative integrators, we consider Dahlquist's test problem of the form

$$\frac{dy}{dx}=\lambda y(x);\ \ y(0)=y_0,\ \ \lambda\in C$$

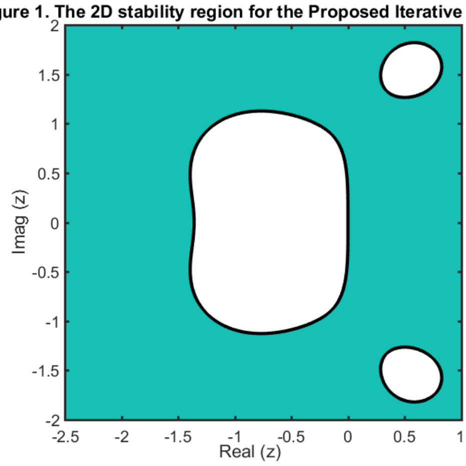
Employing the proposed integrator (7) on this test problem, we obtain the following stability function whose linear stability region is shown by the unshaded region in the Figure 1.

$$k_1=\lambda y_n\ ;\ k_2=\lambda y_n\left[1+\frac{2}{3}h\lambda+h^2\lambda^2\right];\ k_3=\lambda y_n\left[1+\frac{2}{3}h\lambda-\frac{4}{3}h^2\lambda^2+h^3\lambda^3\right]$$

Substituting all of these values in (7), the stability function is found to be of the form:

$$R\left(z\right)=1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{4}z^4 \text{ where } z=h\lambda.$$

Figure 1. The 2D stability region for the Proposed Iterative Integrator



6. NUMERICAL EXPERIMENTS

In this section, some of the linear and nonlinear Cauchy problems in ordinary differential equations have been considered to show the behavior of the developed iterative integrator against other methods from well-established literature having same order of accuracy. Absolute maximum error, absolute error at the last nodal point of the given integration interval and CPU values for time have been presented to observe the performance of the developed method in comparison to other methods. Two standard methods called Runge-Kutta Method with Harmonic Mean of Three Quantities (RK3HM) [16] and Heun's third order method [25] as shown below have been chosen to compare the numerical results obtained through the newly developed iterative integrator.

Heun's Third Order

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0\\
1/3 & 1/3 & 0 & 0\\
2/3 & 0 & 2/3 & 0\\
\hline
 & 1/4 & 0 & 3/4
\end{array}
$$

RK3HM Third Order

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0\\
2/3 & 2/3 & 0 & 0\\
2/3 & -2/3 & 4/3 & 0\\
\hline
 & k_1k_2/(k_1+k_2) & k_2k_3/(k_2+k_3) & 0
\end{array}
$$

Table 1. Errors and CPU time values for Cauchy Problem 1

Problem 1. Nonlinear Cauchy Problem $\frac{dy}{dx}=xy^3-y,\ y(0)=1,\quad y(x)=\frac{2}{\sqrt{2+4x+2e^{2x}}}$			
Step-size/Method	RK3HM	Heun	Proposed
0.1	2.2406e-04	4.3314e-05	6.9364e-06
	2.1363e-04	4.0424e-05	3.8546e-06
	0.0000e+00	0.0000e+00	0.0000e+00
0.05	5.1698e-05	5.2797e-06	6.0548e-07
	4.9489e-05	4.9295e-06	1.4250e-07
	0.0000e+00	1.5625e-02	0.0000e+00
0.025	1.2497e-05	6.4999e-07	6.3439e-08
	1.1988e-05	6.0711e-07	7.4951e-10
	0.0000e+00	0.0000e+00	0.0000e+00
0.0125	3.0773e-06	8.0582e-08	7.2203e-09
	2.9553e-06	7.5283e-08	8.8340e-10
	1.5625e-02	1.5625e-02	1.5625e-02

Figure 2. Absolute Errors for Problem 1 with h = 0.1

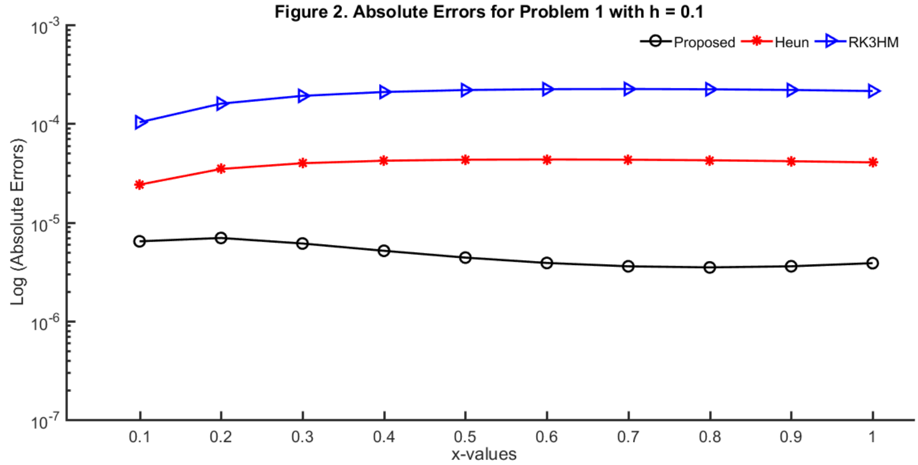


Table 2. Errors and CPU time values for Cauchy Problem 2

Problem2. Linear Cauchy Problem $\frac{dy}{dx}=x-y,\quad y(0)=1,\quad y(x)=x+2e^{-x}-1$			
Step-size/Method	RK3HM	Heun	Proposed
0.1	2.0501e-03	3.3214e-05	3.7347e-05
	3.1498e-04	3.3214e-05	4.0107e-06
	0.0000e+00	0.0000e+00	0.0000e+00
0.05	6.1115e-04	3.9886e-06	4.1479e-06
	2.1978e-06	3.9886e-06	6.8968e-08
	0.0000e+00	0.0000e+00	0.0000e+00
0.025	1.9997e-04	4.8869e-07	4.8762e-07
	3.7473e-05	4.8869e-07	4.2089e-08
	0.0000e+00	0.0000e+00	0.0000e+00
0.0125	1.0845e-04	6.0478e-08	5.9073e-08
	5.3348e-05	6.0478e-08	7.2876e-09
	1.5625e-02	1.5625e-02	0.0000e+00

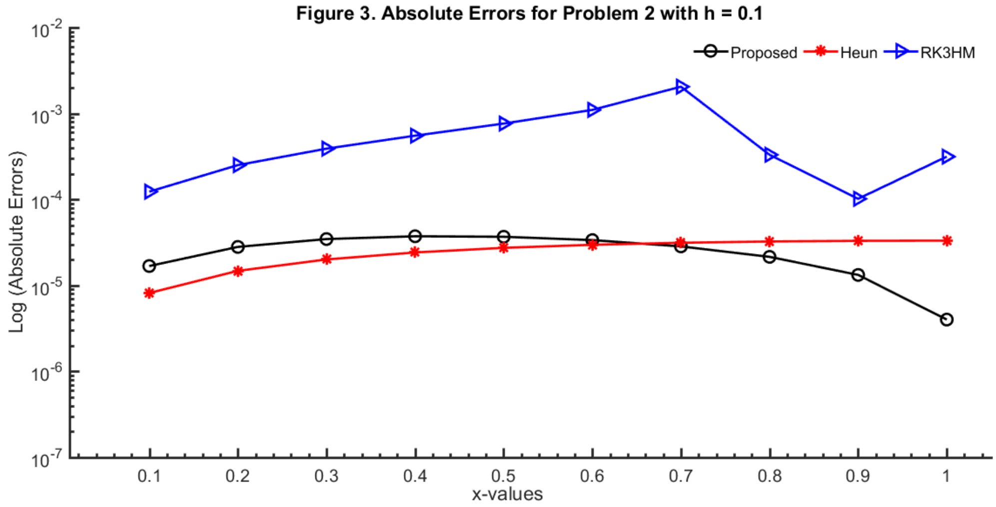


Table 3. Errors and CPU values for Cauchy Problem 3

Problem 3. Nonlinear Cauchy Problem $\frac{dy}{dx}=\frac{x^2}{y},\quad y(0)=1,\quad y(x)=\sqrt{2\left(\frac{x^3}{3}+\frac{1}{2}\right)}$			
Step-size/Method	**RK3HM**	**Heun**	**Proposed**
0.1	1.5241e-03	8.3753e-06	3.5068e-06
	1.5241e-03	7.7843e-06	1.6030e-06
	0.0000e+00	0.0000e+00	0.0000e+00
0.05	3.8551e-04	1.0250e-06	3.6291e-07
	3.8551e-04	9.5083e-07	1.6466e-07
	0.0000e+00	0.0000e+00	0.0000e+00
0.025	9.7000e-05	1.2731e-07	4.5498e-08
	9.7000e-05	1.1762e-07	4.5498e-08
	0.0000e+00	0.0000e+00	0.0000e+00
0.0125	2.4332e-05	1.5843e-08	7.3060e-09
	2.4332e-05	1.4631e-08	7.3060e-09
	1.5625e-02	0.0000e+00	0.0000e+00

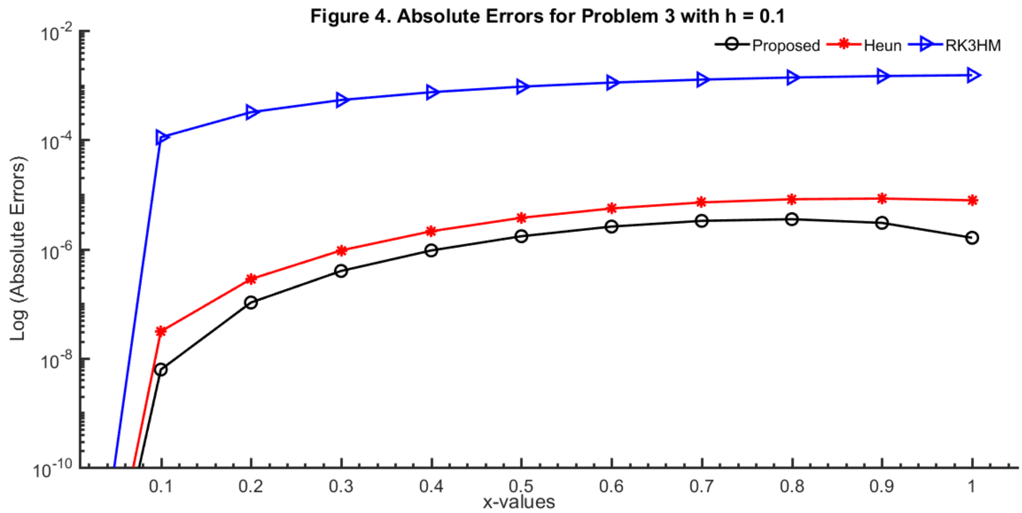
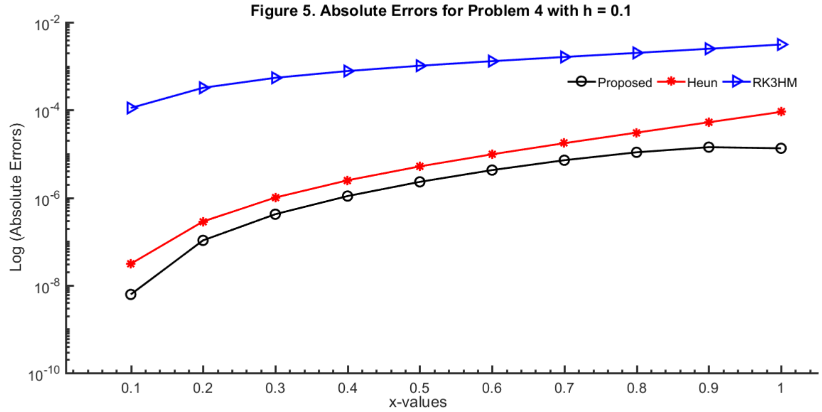


Table 4. Errors and CPU time values for Cauchy Problem 4

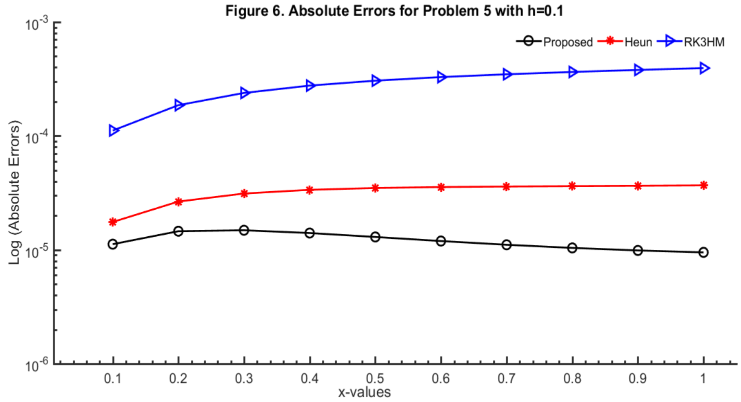
Problem 4. Linear Cauchy Problem $\frac{dy}{dx}=x^2y,\quad y(0)=1,\quad y(x)=e^{\frac{x^3}{3}}$			
Step-size/Method	**RK3HM**	**Heun**	**Proposed**
0.1	3.1171e-03	9.0292e-05	1.4165e-05
	3.1171e-03	9.0292e-05	1.3359e-05
	1.5625e-02	0.0000e+00	0.0000e+00
0.05	8.2411e-04	1.1713e-05	1.6414e-06
	8.2411e-04	1.1713e-05	1.2474e-06
	0.0000e+00	0.0000e+00	0.0000e+00
0.025	2.1194e-04	1.4897e-06	1.9677e-07
	2.1194e-04	1.4897e-06	1.2857e-07
	1.5625e-02	0.0000e+00	0.0000e+00
0.0125	5.3744e-05	1.8777e-07	2.4068e-08
	5.3744e-05	1.8777e-07	1.4347e-08
	0.0000e+00	1.5625e-02	1.5625e-02

Figure 5. Absolute Errors for Problem 4 with h = 0.1



Problem 5. Nonlinear Cauchy Problem $\frac{dy}{dx}=xy^2-y,\quad y(0)=1,\quad y(x)=\frac{1}{x+1}$			
Step-size/Method	**RK3HM**	**Heun**	**Proposed**
0.1	3.9431e-04	3.6815e-05	1.4849e-05
	3.9431e-04	3.6815e-05	9.4940e-06
	0.0000e+00	0.0000e+00	0.0000e+00
0.05	9.2396e-05	4.4832e-06	1.5897e-06
	9.2396e-05	4.4832e-06	9.2296e-07
	0.0000e+00	0.0000e+00	0.0000e+00
0.025	2.2447e-05	5.5229e-07	1.8282e-07
	2.2447e-05	5.5229e-07	1.0023e-07
	0.0000e+00	1.5625e-02	0.0000e+00
0.0125	5.5374e-06	6.8510e-08	2.1895e-08
	5.5374e-06	6.8510e-08	1.1612e-08
	1.5625e-02	1.5625e-02	1.5625e-02

Figure 6. Absolute Errors for Problem 5 with h=0.1



7. RESULTS AND DISCUSSIONS

The newly developed third order iterative integrator is capable of solving Cauchy problems in the field of computational and applied mathematics. The maximum error and last error with step sizes 0.1, 0.05, 0.025 and 0.0125 are tabulated along-with the values of CPU timing in seconds. One may observe from these tabulated data that the absolute maximum and last error produced by the proposed iterative integrator are much smaller than the errors produced by other methods having same order of accuracy while consuming same amount of time on average. Similar sort of behavior has been observed while taking the step-size as large as 0.1 as shown by the Figures 2-6 for all the iterative integrators under consideration. The numerical results obtained through the proposed iterative integrator produce numerical values approximately close to the exact solution in comparison to the values obtained through Runge-Kutta Method with Harmonic Mean of Three Quantities and Heun's third order method. For the proposed iterative integrator, small step size is also enough in comparison for other methods as shown in the Tables and the Figures above. Finally, it has been observed that the proposed iterative integrator is converging faster than the RK3HM and Heun's third order method and it is the most effective integrator for solving the Cauchy problems in ordinary differential equations as long as it is compared with the iterative integrators having same order of local accuracy as that of the proposed iterative integrator.

8. CONCLUSION

This paper develops a new single step Runge-Kutta iterative integrator for solving Cauchy problems in ordinary differential equations. The integrator is found to be third order accurate and explicit in nature. Its linear stability analysis gives the stability region which proves conditional stability of the proposed integrator. Examples in this paper proved that it is more accurate and effective integrator than some existing standard methods. Tables 1 to 5 above show the maximum error, the last error and CPU times related to all the integrators under consideration for the Cauchy problems with the variation in the step size. In addition, absolute errors produced by the above iterative integrators are smallest in case of the proposed integrator as shown by the Figures 2-6. The computations above evidently display the better accuracy of the integrator. The Runge-Kutta Method with Harmonic Mean grows faster in error than third order Heun and the proposed one. Hence, the proposed integrator performs best among the integrators taken for comparison. Based on the five Cauchy problems solved above, it follows that the proposed integrator is quite efficient specifically in terms of local accuracy. It can be concluded that the proposed integrator is powerful and effective in finding numerical solutions Cauchy type problems arising frequently in the field of computational and applied mathematics.

ACKNOWLEDGEMENT

The authors are highly thankful to the anonymous reviewers for their constructive criticism and suggestions to improve the first version of this research paper.

MATLAB CODE

```
% A New Third Order Iterative Integrator for Cauchy Problems
% Numerical Problem 1.y'=xy^3-y, y(0)=1;
% Exact Solution: y(x)=2/sqrt(2+4*x+2*exp(2*x));  Partial Derivative f_y=3*x*y^2-1
% Integration Interval [0,1]
clc; clear; close all; format shorte
x(1)=0; y(1)=1; h=0.1; xfinal=1; N=ceil((xfinal-x(1))/h);
Time_Proposed=cputime;
fori=1:N
   k1=f_der(x(i),y(i));
   k2=f_der(x(i)+(2*h/3),y(i)+(2*h/3)*k1+(h^2)*k1*(3*x(i)*y(i)^2-1));
   k3=f_der(x(i)+(2*h/3),y(i)-h*(k1/3-k2)-2*(h^2)*k1*(3*x(i)*y(i)^2-1));
y(i+1)=y(i)+(h/4)*(k1+2*k2+k3);
x(i+1)=x(i)+h;
end
Time_Daud=cputime-Time_Proposed;
t=x(1):h:xfinal;
Exact=2./sqrt(2+4*t+2*exp(2*t));
Error_Daud=abs(Exact-y);
Err_Max_Daud=max(Error_Daud);
Err_Last_Daud=abs(Exact(length(t))-y(length(t)));
semilogy(t,Error_Daud,'ko-'), hold on
%%
```

```
% Third Order Heun Method
TIME=cputime;
fori=1:N
    k1=f_der(x(i),y(i));
   k2=f_der(x(i)+h*(1/3),y(i)+(1/3)*h*k1);
   k3=f_der(x(i)+(2/3)*h,y(i)+(2/3)*h*k2);
y(i+1)=y(i)+h*(1/4)*(k1+3*k3);
x(i+1)=x(i)+h;
end
Time_Heun3=cputime-TIME;
Error_Heun3=abs(Exact-y);
Err_Max_Heun3=max(Error_Heun3);
Err_Last_Heun3=abs(Exact(length(x))-y(length(x)));
semilogy(t,Error_Heun3,'r*-')
%%
%3rd order MODIFIED RK Rule using Harmonic Mean
RK3HM=cputime;
fori=1:N
   k1=f_der(x(i),y(i));
   k2=f_der(x(i)+(2/3)*h,y(i)+(2/3)*h*k1);
   k3=f_der(x(i)+(2/3)*h,y(i)-h*(2/3)*k1+h*(4/3)*k2);
   y(i+1)=y(i)+h*((k1*k2)/(k1+k2)+(k2*k3)/(k2+k3));
x(i+1)=x(i)+h;
end
Time_RK3HM=cputime-RK3HM;
Error_RK3HM=abs(Exact-y);
Err_Max_RK3HM=max(Error_RK3HM);
Err_Last_RK3HM=abs(Exact(length(x))-y(length(x)));
semilogy(t,Error_RK3HM,'b>-')
%%
functiondydx=f_der(x,y)
dydx=x.*y.^3-y;
end
```

REFERENCES

[1] Lambert, John D. "Computational methods in ordinary differential equations." (1973).

[2] Dory, Robert A. "Ordinary Differential Equations." *Computers in Physics* 3.5 (1989): 88-91.

[3] Nieto, Juan J., and Rosana Rodríguez-López. "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations." *Order* 22.3 (2005): 223-239.

[4] Chicone, Carmen. *Ordinary differential equations with applications*. Vol. 34. Springer Science & Business Media, 2006.

[5] Immler, Fabian, and Johannes Hölzl. "Numerical Analysis of Ordinary Differential Equations." (2013).

[6] Butcher, John Charles. *Numerical methods for ordinary differential equations*. John Wiley & Sons, 2016.

[7] Gear, C. William. "The automatic integration of ordinary differential equations." *Communications of the ACM* 14.3 (1971): 176-179.

[8] Jordan, Dominic William, and Peter Smith. *Nonlinear ordinary differential equations: an introduction to dynamical systems*. Vol. 2. Oxford University Press, USA, 1999.

[9] Teschl, Gerald. *Ordinary differential equations and dynamical systems*. Vol. 140. Providence: American Mathematical Society, 2012.

[10] Wusu, Ashiribo Senapon, Moses Adebowale Akanbi, and Solomon Adebola Okunuga. "A three-stage multiderivative explicit Runge-Kutta method." *American Journal of Computational Mathematics* 3.02 (2013): 121.

[11] Akanbi, Moses Adebowale. "On 3-stage geometric explicit Runge–Kutta method for singular autonomous initial value problems in ordinary differential equations." *Computing* 92.3 (2011): 243-263.

[12] Gadisa, Gashu, and Habtamu Garoma. "Comparison of higher order Taylor's method and Runge-Kutta methods for solving first order ordinary differential equations." *Journal of Computer and Mathematical Sciences* 8.1 (2017): 12-23.

[13] SO, Ayinde, and E. A. Ibijola. "A new numerical method for solving first order differential equations." *American Journal of Applied Mathematics and Statistics* 3.4 (2015): 156-160.

[14] Rabiei, F. A. R. A. N. A. K., Fudziah Ismail, and Mohamed Suleiman. "Improved Runge-Kutta Methods for Solving Ordinary Differential Equations." *Sains Malaysiana* 42.11 (2013): 1679-1687.

[15] Ahmad, R. R., and N. Yaacob. "Third-order composite Runge–Kutta method for stiff problems." *International Journal of Computer Mathematics* 82.10 (2005): 1221-1226.

[16] Wazwaz, Abdul-Majid. "A modified third order Runge-Kutta method." *Applied Mathematics Letters* 3.3 (1990): 123-125.

[17] Wusu, Ashiribo Senapon, Moses Adebowale Akanbi, and Bakre Omolara Fatimah. "On the Derivation and Implementation of a Four Stage Harmonic Explicit Runge-Kutta Method." *Applied Mathematics* 6.04 (2015): 694.

[18] Goeken, David, and Olin Johnson. "Runge–Kutta with higher order derivative approximations." *Applied numerical mathematics* 34.2-3 (2000): 207-218.

[19] Ismail, Fudziah. "Third-order Improved Runge-Kutta method for solving ordinary differential equation." *International Journal of Applied Physics and Mathematics* 1.3 (2011): 191.

[20] Ramos, Higinio. "A non-standard explicit integration scheme for initial-value problems." *Applied Mathematics and Computation* 189.1 (2007): 710-718.

[21] Ramos, Higinio. "A Nonlinear Explicit One-Step Integration Scheme for Singular Autonomous Initial Value Problems." *AIP Conference Proceedings*. Vol. 936. No. 1. AIP, 2007.

[22] Ramos, Higinio, and Jesús Vigo-Aguiar. "A new algorithm appropriate for solving singular and singularly perturbed autonomous initial-value problems." *International Journal of Computer Mathematics* 85.3-4 (2008): 603-611.

[23] Ramos, Higinio, et al. "Numerical solution of nonlinear singularly perturbed problems on non uniform meshes by using a non-standard algorithm." *Journal of mathematical chemistry*48.1 (2010): 38-54.

[24] Qureshi, S., A. A. Shaikh, and M. S. Chandio. "Critical Study of a Nonlinear Numerical Scheme for Initial Value Problems." *Sindh University Research Journal-SURJ (Science Series)* 47.4 (2015).

[25] Burden, Richard L., and J. Douglas Faires. "Numerical analysis. 2001." *Brooks/Cole, USA* (2001).