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On Uni-Soft Ideals and Uni-Soft Interior Ideals of Ordered Semigroups

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ABSTRACT

In this paper, we define and investigate some properties of uni-soft ideals and uni-soft interior ideals of ordered semigroups. Furthermore, we prove that in regular ordered semigroups the uni-soft ideals and the uni-soft interior ideals coincide. Finally, we introduce the concept of uni-soft simple ordered semigroups and show that in a simple ordered semigroup every uni-soft interior ideal is a constant function. Using the notion of uni-soft left (right) ideals, characterizations of a left (right) simple semigroup are provided.
KEYWORDS: Ordered semigroups, Soft sets, uni-soft interior ideal, simple ordered semigroup, regular (intra-regular) ordered semigroup, characteristic soft set.

1. INTRODUCTION

Molodstov [1] pointed out that the important existing theories viz. Probability theory, Fuzzy set theory [2], Intuitionistic Fuzzy set theory [3], rough set theory [4] etc, which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. He further pointed out that the reason for these difficulties is the inadequacy of parameterization tools of the theory. In 1999 he initiated the novel concept of soft set theory to deal with the uncertainty associated with the data, whereas on the other hand it has the ability to represent the data in a useful manner. In soft set theory, the problem of setting the membership function does not arises, which make the theory easily applied to many different fields including game theory, Riemann integration, operation research, perron integration, theory of measurement and so on. At present work, soft set theory and its applications are progressing rapidly. After Molodstov work, some operations and applications of soft sets were studied by many researchers including Ali et al. [5], Aktas and Cağman [6], Chen et al. [7], and Maji et al. [8]. Recently, Neog and Sut [9, 10] have studied the notion of fuzzy soft union, fuzzy soft intersection, complement of a fuzzy set and several other properties of fuzzy soft sets along with examples and proofs of certain results. Jun *et al.*, [11] applied the concept of soft set theory to ordered semigroups. They applied the notion of soft sets by Molodtsov to ordered semigroups and introduced the notions of (trivial, whole) soft ordered semigroups, soft ordered sub semigroups, soft r-ideals, soft l-ideals, and r-idealistic and l-idealistic soft ordered semigroups. They investigated various related properties by using these notions. In [18, 19, 20] Khan *et al*., characterized different classes of ordered semigroups by using uni-soft quasi-ideals and uni-soft ideals. In this paper the concept of a uni-soft interior ideal is introduced and it is given that in regular and intra regular ordered semigroups, the concepts of uni-soft ideals and uni-soft interior ideals are coincide. We also prove that in a simple ordered semigroup every uni-soft interior ideal is a constant function. Using the notion of uni-soft left (right) ideals, characterizations of a left (right) simple semigroup are provided. It is shown that if A is an interior ideal of S if and only if $(\varepsilon,\delta)-$ characteristic soft set $\left(\chi_A^{(\varepsilon,\delta)}, S\right)$ over U is a uni-soft interior ideal over U.

2. Basic definitions and preliminaries

By an ordered semigroup $(\text{or } po\text{-}semigroup)$ we mean a structure $(S,\cdot,\leq)$ in which $(S,\cdot)$ is a semigroup, $(S,\leq)$ is a poset and for all $a,b,x\in S$, the condition $a\leq b\Rightarrow xa\leq xb$ and $ax\leq bx$ is satsfied.

For subsets A and B of an ordered semigroup S, we denote

$$AB := \{ab \mid a\in A, b\in B\}.$$

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If $A \subseteq S$, we denote $(A] := \{t \in S \mid t \le h$ for some $h \in A\}$. For $a \in S$, we write $(a]$ instead of $(\{a\}]$. For subsets A and B of an ordered semigroup S, we have $A \subseteq (A]$. If $A \subseteq B$, then $(A] \subseteq (B]$, $(A](B] \subseteq (AB]$, $((A]] = (A]$ and $((A](B]] \subseteq (AB]$.

Let $(S,\cdot,\le)$ be an ordered semigroup. A non-empty subset of S is called a *subsemigroup* of S if $A^2 \subseteq A$.

A non-empty subset of S is called a *right* (resp., *left*) *ideal* of S if:

(1) $AS \subseteq A$ (resp., $SA \subseteq A$) and

(2) if $a \in A$ and $S \ni b \le a$, then $b \in A$.

If A is both a right and a left ideal of S, then it is called an ideal of S.

A non-empty subset of S is called a*interiorideal* of S if:

(1) $SAS \subseteq A$,

(2) if $a \in A$ and $S \ni b \le a$, then $b \in A$.

An ordered semigroup S is said to be regular if for every $x \in S$, there exist $a \in S$ such that $a \le axa.$.

An ordered semigroup S is called intra-regular if for every $a \in S$, there exist $x, y \in S$ such that $a \le xa^2 y$.

An ordered semigroup S is called semi simple if for every $a \in S$, there exist $x, y, z \in S$ such that $a \le xayaz$.

An ordered semigroup S is said to be left (resp., right) simple if it contains no proper left (resp., right) ideal.

An ordered semigroup is said to be simple if it contains no proper two-sided ideal.

From now on, U be an initial universe set, E be a set of parameters, $P(U)$ denotes the power set of U and $A, B, C... \subseteq E$.

A soft set theory is introduced by Çağman [6] provided new definitions and various results on soft set theory.

A pair (f_A, E) is called a soft set over U if and only if f_A is a mapping of E into the set of all subsets of the set U. e, g

$$f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\},$$

where $f_A : E \to P(U)$ such that $f_A(x) = \phi$ if $x \notin A$. The function f_A is also called an*approximation function of the soft* (f_A, E). It is clear that a soft set is a *parameterized family* of subsets of U. Note that the set of all soft sets over U will be denoted $S(U)$ and we take $E = S$.

For a soft (f_A, S) over U and $\delta \subseteq U$. The $\delta-$exclusive set of (f_A, S), denoted by $e_A(f_A; \delta)$, is defined as

$$e_A(f_A; \delta) = \{x \in A \mid f_A(x) \subseteq \delta\}.$$

For any soft sets (f_A, S) and (f_B, S) over U, we define

$$(f_A, S) \,\tilde{\subseteq}\, (f_B, S) \text{ if } f_A(x) \subseteq f_B(x), \forall x \in S$$

The *soft union* of (f_A, S) and (f_B, S), denoted by $(f_A, S) \,\tilde{\cup}\, (f_B, S) = (f_{A \tilde{\cup} B}, S)$ is defined by

$$(f_A \,\tilde{\cup}\, f_B)(x) = f_A(x) \cup f_B(x), \forall x \in S.$$

The *soft intersection* of (f_A, S) and (f_B, S), denoted by $(f_A, S) \,\tilde{\cap}\, (f_B, S) = (f_{A \tilde{\cap} B}, S)$ is defined by

$$(f_A \,\tilde{\cap}\, f_B)(x) = f_A(x) \cap f_B(x), \forall x \in S.$$

For a non-empty subset A of S, the *characteristic soft set* (χ_A, S) over U is a soft set defined as follows:

$$\chi_A : S \to P(U), x \mapsto \begin{cases} U & \text{if } x \in A, \\ \phi & \text{if } x \in S \setminus A. \end{cases}$$

Let (f_S, S) and (g_S, S) be uni-soft set over U. The uni-soft product of (f_S, S) and (g_S, S) is defined to be the soft $(f_S \Diamond g_S, S)$ over U in which $f_S \Diamond g_S$ is a mapping from S to $P(U)$ given by

$$(f_S \Diamond g_S)(x) = \begin{cases} \bigcap_{x \le yz} \{f_S(y) \cup g_S(z)\} & \text{if } x \le yz \text{ for some } y, z \in S, \\ U & \text{otherwise.} \end{cases}$$

Definition 1.*(see [11])* Let S be an ordered semigroup. A soft set (f_S, S) of S over U is called a *uni-soft semigroup* of S over U if:

$$f_S(xy) \subseteq f_S(x) \cup f_S(y) \text{ for all } x, y \in S.$$

Definition 2.*(see [11]*) Let $(S,\cdot,\leq)$ be an ordered semigroup. A soft set (f_S, S) of S over U is called a *uni-soft left* (resp., *right*) *ideal* of S over U if:

(1) $x \leq y \Rightarrow f_S(x) \subseteq f_S(y)$,

(2) $f_S(xy) \subseteq f_S(y)$ (resp., $f_S(xy) \subseteq f_S(x)$) for all $x, y \in S$.

If (f_S, S) is both a uni-soft left ideal and a uni-soft right ideal over U, then (f_S, S) is called a *uni-soft two-sided ideal* over U.

3. Uni-soft ideals and uni-soft interior ideals of ordered semigroups

In this section, we first introduce the concepts of uni-soft interior ideals of ordered semigroup. And And then show that every uni-soft ideal is a uni-soft interior ideal.

Definition3.Let $(S,\cdot,\leq)$ be an ordered semigroup. A soft (f_S, S) over U is called uni-soft interior ideal over U if:

(1) $x \leq y \Rightarrow f_S(x) \subseteq f_S(y)$,

(2) $f_S(xay) \subseteq f_S(a)$ for all $a, x, y \in S$.

Example 1.[19] *Let* $S = \{a, b, c, d\}$ *be an ordered semigroup with the following multiplication table and order relation which are given in the following:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,d)\}.$$

Let (f_S, S) be a soft set over $U = \mathrm{Z}$ defined by

$$f_S : S \to P(U),\ x \mapsto \begin{cases} 6\mathrm{N} & \text{if } x = a, \\ 3\mathrm{Z} & \text{if } x \in \{b, d\}, \\ 3\mathrm{N} & \text{if } x = c. \end{cases}$$

Then $f_S(xyz) = f_S(a) = 6\mathrm{N} \subseteq f_S(y)$ for every $x, y, z \in S$. Therefore, (f_S, S) is a uni-soft inteior ideal over U.

Example 2.Consider the ordered semigroup in Example 1. Let (f_S, S) be a soft set over $U = D_2 = \{< x, y >:\ x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ defined by

$$f_S : S \to P(U),\ x \mapsto f_S(x) = \begin{cases} \{e\} & \text{if } x = a, \\ \{e, x, y\} & \text{if } x \in \{b, d\}, \\ \{e, x\} & \text{if } x = c. \end{cases}$$

Then $f_S(xyz) = f_S(a) = \{e\} \subseteq f_S(y)$ for every $x, y, z \in S$. Therefore, (f_S, S) is a uni-soft inteior ideal over U.

Lemma 1.Let $(S,\cdot,\leq)$ be an ordered semigroup. Then every uni-soft two-sided ideal over U is a uni-soft interior ideal over U.

Proof. Let (f_S, S) be an uni soft two-sided ideal over U and $x, y, a \in S$. Then,

$$f_S(xay) = f_S((xa)y) \subseteq f_S(xa) \subseteq f_S(a).$$

Let $x, y \in S$ be such that $x \leq y$. Then $f_S(x) \subseteq f_S(y)$, because (f_S, S) is a uni-soft two-sided ideal over U. Thus (f_S, S) is a uni-soft interior ideal over U.

As it is well-known that every uni-soft two-sided ideal is a uni-soft interior ideal, but the converse is not true in general.

The uni-soft interior ideal (f_S, S) over $U = Z$ in Example 1, is not a uni-soft left ideal over U, since $f_S(dc) = f_S(b) = 3Z \not\subset 3N = f_S(c)$ and hence it is not a uni-soft two-sided ideal over $U = Z$.

Example 1. *Consider the ordered semigroup in Example* 1. *Define a soft set* (f_S, S) *over* $U = \{1,2,3,4,5,6,7,8,9,10\}$ *such that*

$$f_S : S \to P(U),\ x \mapsto f_S(x) = \begin{cases} \{2\} & \text{if } x = a, \\ \{1,2,3\} & \text{if } x \in \{b,d\}, \\ \{1,2\} & \text{if } x = c. \end{cases}$$

Then $f_S(xyz) = f_S(a) = \{2\} \subseteq f_S(y)$ for every $x, y, z \in S$. Therefore, (f_S, S) is a uni-soft inteior ideal over U.

Consider the uni-soft interior ideal (f_S, S) over $U = \{1,2,3,4,5,6,7,8,9,10\}$ in Example 2, is not a uni-soft left ideal over U, since $f_S(dc) = f_S(b) = \{1,2,3,\} \not\subset \{1,2\} = f_S(c)$ and hence it is not a uni-soft two-sided ideal over U.

In the following result we provide a condition for a uni-soft interior ideal over U to be a uni-soft two-sided ideal over U.

Theorem 1.Let $(S,\cdot,\leq)$ be a regular ordered semigroup. Then every uni-soft interior ideal over U is a uni-soft two-sided ideal over U.

Proof. Let (f_S, S) is a uni-soft interior ideal over U and let $x, y \in S$. Since S is a regular, then there exist $a, b \in S$ such that $x \leq xax$ and $y \leq yby$. we have

$$f_S(xy) \subseteq f_S((xax)y) = f_S((xa)xy) \subseteq f_S(x),$$

and

$$f_S(xy) \subseteq f_S(x(yby)) = f_S(xy(by)) \subseteq f_S(y).$$

Now let $x, y \in S$ be such that $x \leq y$. Then $f_S(x) \subseteq f_S(y)$, because (f_S, S) is a uni-soft interior ideal of S over U. Therefore (f_S, S) is a uni-soft two-sided ideal over U.

By Lemma 1 and Theorem 1 we have the following:

Remark 1.In regular ordered semigroups, the concepts of uni-soft two-sided ideals and uni-soft interior ideals coincide.

Theorem 2.Let $(S,\cdot,\leq)$ be an intra-regular ordered semigroup, (f_S, S) is a uni-soft interior ideal over U. Then (f_S, S) is a uni-soft two-sided ideal over U.

Proof.Let (f_S, S) is a uni-soft interior ideal over U. Let $x, y \in S$, since S is a intra-regular then there exist $a, b \in S$ such that $x \leq ax^2a$ and $y \leq by^2b$. Since (f_S, S) is an uni-soft interior ideal of S, we have

$$f_S(xy) \subseteq f_S((ax^2a)y) = f_S((ax)x(ay)) \subseteq f_S(x),$$

and

$$f_S(xy) \subseteq f_S(x(by^2b)) = f_S((xb)y(yb)) \subseteq f_S(y).$$

Let $x, y \in S$ be such that $x \leq y$. Then $f_S(x) \subseteq f_S(y)$, because (f_S, S) is a uni-soft interior ideal of S over U. Thus (f_S, S) is a uni-soft tw0-sided ideal of S over U.

By Lemma 1 and Theorem 2 we have the following:

Remark2.In intra regular ordered semigroups, the concepts of uni-soft two-sided ideals and uni-soft interior ideals coincide.

Theorem 3. Let $(S,\cdot,\leq)$ be an ordered semigroup with identity e. Then every uni-soft interior ideal over U is a uni-soft two sided ideal over U.

Proof. Let (f_S, S) is a uni-soft interior ideal over U and $x, y \in S$. Then $f_S(xy) = f_S(xye) \subseteq f_S(y)$ and $f_S(xy) = f_S(exy) \subseteq f_S(x)$. Furthermore let $x, y \in S$ be such that $x \leq y$. Then $f_S(x) \subseteq f_S(y)$, because (f_S, S) is a uni-soft interior ideal of S over U. Therefore, (f_S, S) is a uni-soft two-sided ideal over U.

Proposition 1.(see *[17]) Let $(S,\cdot,\leq)$ be a semi simple ordered semigroup, (f_S, S) a uni-soft interior ideal of S over U. Then (f_S, S) is a uni-soft two-sided ideal of S over U.*

By Lemma 1 and Proposition 1 we have the following:

Remark 3.*In semisimple ordered semigroups, the concepts of uni-soft ideals and uni-soft interior ideals coincide.*

For a non-empty subset A of S and $\varepsilon, \delta \in P(U)$ with $\varepsilon \not\subset \delta$, define a soft set $\chi_A^{(\varepsilon,\delta)}$ as follows:

$$\chi_A^{(\varepsilon,\delta)} : S \to, x \mapsto \begin{cases} \varepsilon & \text{if } x \in A, \\ \delta & \text{otherwise.} \end{cases}$$

The soft set $(\chi_A^{(\varepsilon,\delta)}, S)$ is called $(\varepsilon,\delta)-$ characteristic uni-soft set over U. The soft set $(\chi_A^{(\varepsilon,\delta)}, S)$ is called $(\varepsilon,\delta)-$ identity uni-soft set over U. The $(\varepsilon,\delta)-$ characteristic uni-soft set $(\chi_A^{(\varepsilon,\delta)}, S)$ with $\varepsilon = \phi$ and $\delta = U$ is called the characteristic soft set, and is denoted by (χ_A^c, S). The $(\varepsilon,\delta)-$ identity uni-soft set with $\varepsilon = \phi$ and $\delta = \phi$ is called the empty soft set, and is denoted by (χ_S^c, S).

Theorem 4. Let $(S,\cdot,\leq)$be an ordered semigroup. Then for a non-empty subset A of S, the following conditions are equivalent:

(1) A is an interior ideal of S.

(2) The $(\varepsilon,\delta)-$ characteristic soft set $(\chi_A^{(\varepsilon,\delta)}, S)$ over U is a uni-soft interior ideal over U for any $\varepsilon, \delta \in P(U)$with $\varepsilon \not\subset \delta$.

Proof. Suppose that A is an interior ideal of S and $x, y, z \in S$. Let $\varepsilon, \delta \in P(U)$ with $\varepsilon \not\subset \delta$, if $y \notin A$ implies that $\chi_A^{(\varepsilon,\delta)}(y) = \delta$, then either $xyz \in A$ or $xyz \notin A$ implies that

$$\chi_A^{(\varepsilon,\delta)}(xyz) \subseteq \delta = \chi_A^{(\varepsilon,\delta)}(y).$$

If $y \in A$, $\chi_A^{(\varepsilon,\delta)}(y) = \varepsilon$ and $xyz \in SAS \subseteq A$. Hence

$$\chi_A^{(\varepsilon,\delta)}(xyz) = \varepsilon = \chi_A^{(\varepsilon,\delta)}(y).$$

Now let $x, y \in S$ be such that $x \leq y$. If $y \in A$ then $\chi_A^{(\varepsilon,\delta)}(y) = \varepsilon$. Since A is a interior ideal of S, so $x \in A$. Hence $\chi_A^{(\varepsilon,\delta)}(x) = \varepsilon = \chi_A^{(\varepsilon,\delta)}(y)$. If $y \notin A$ then $\chi_A^{(\varepsilon,\delta)}(y) = \delta$, then obviously, $\chi_A^{(\varepsilon,\delta)}(x) \subseteq \delta = \chi_A^{(\varepsilon,\delta)}(y)$. Therefore, $(\chi_A^{(\varepsilon,\delta)}, S)$ is a uni-soft interior ideal over U for any $\varepsilon, \delta \in P(U)$ with $\varepsilon \not\subset \delta$.

Conversely, suppose that $(\varepsilon,\delta)-$ characteristic soft set $(\chi_A^{(\varepsilon,\delta)}, S)$ over U is a uni-soft interior ideal over U for any $\varepsilon, \delta \in P(U)$ with $\varepsilon \not\subset \delta$. Let a be any element of SAS. Then $a = xyz$ for some $x, z \in S$ and $y \in A$. Then

$$\chi_A^{(\varepsilon,\delta)}(a) = \chi_A^{(\varepsilon,\delta)}(xyz) \subseteq \chi_A^{(\varepsilon,\delta)}(y) = \varepsilon,$$

and so $\chi_A^{(\varepsilon,\delta)}(a) = \varepsilon$. Thus $a \in A$, which means that $SAS \subseteq A$.

Furthermore, let $x, y \in S$ with $x \leq y \in A$. Then $\chi_A^{(\varepsilon,\delta)}(y) = \varepsilon$. By hypothesis we have

$$\chi_A^{(\varepsilon,\delta)}(x) \subseteq \chi_A^{(\varepsilon,\delta)}(y) = \varepsilon,$$

which means that $\chi_A^{(\varepsilon,\delta)}(x)=\varepsilon.$ Hence $x\in A.$ Therefore A is an interior ideal of $S.$

Theorem 5. Let $(S,\cdot,\leq)$ be an ordered semigroup. Then for a non-empty subset A of $S,$ the following conditions are equivalent:

(1) A is an interior ideal of $S.$

(2) The characteristic soft set (χ_A^c,S) over U is a uni-soft interior ideal over U.

Proof.The proof follows from Theorem 4.

Theorem 6. Let $(S,\cdot,\leq)$ be an ordered semigroup and (f_S,S) a soft set of S over U. Then (f_S,S) is a uni-soft uni-soft interior ideal over U if and only if satisfies that

(1) $(\chi_S^c\Diamond f_S\Diamond\chi_S^c,S)\mathrel{\tilde{\supseteq}}(f_S,S),$

(2) $x\leq y\Rightarrow f_S(x)\subseteq f_S(y),\forall x,y\in S.$

Proof. Suppose that (f_S,S) is a uni-soft interior ideal over $U.$ Let x be any element of S and there exist $y,z,p,q\in S.$ If x is not expressed as $x\leq yz$, then $(\chi_S^c\Diamond f_S\Diamond\chi_S^c)(x)=U\supseteq(f_S)(x).$ Assume that x is expressed as $x\leq yz$ and $y\leq pq,$ then

$$\begin{aligned}(\chi_S^c\Diamond f_S\Diamond\chi_S^c)(x)&=\bigcap_{x\leq yz}\{(\chi_S^c\Diamond f_S)(y)\cup\chi_S^c(z)\}\\&=\bigcap_{x\leq yz}\left\{\left(\bigcap_{y\leq pq}\{\chi_S^c(p)\cup f_S(q)\}\right)\cup\chi_S^c(z)\right\}\\&\supseteq\bigcap_{x\leq yz}\left\{\left(\bigcap_{y\leq pq}\{\phi\cup f_S(q)\}\right)\cup\phi\right\}\\&\supseteq\bigcap_{x\leq pqz}\{f_S(q)\cup\phi\}\\&\supseteq f_S(x),\end{aligned}$$

since (f_S,S) is a uni-soft interior ideal over $U.$ So $f_S(x)\subseteq f_S(pqz)\subseteq f_S(q).$ Therefore, $(\chi_S^c\Diamond f_S\Diamond\chi_S^c,S)\mathrel{\tilde{\supseteq}}(f_S,S).$

Conversely, assume that $(\chi_S^c\Diamond f_S\Diamond\chi_S^c,S)\mathrel{\tilde{\supseteq}}(f_S,S).$ For any $a,x,y\in S,$ we have

$$\begin{aligned}f_S(xay)&\subseteq(\chi_S^c\Diamond f_S\Diamond\chi_S^c)(xay)\\&=\bigcap_{xay\leq pq}\{(\chi_S^c\Diamond f_S)(p)\cup\chi_S^c(q)\}\\&\subseteq(\chi_S^c\Diamond f_S)(xa)\cup\chi_S^c(y)\\&=\left(\bigcap_{xa\leq pq}\{\chi_S^c(p)\cup f_S(q)\}\right)\cup\phi\\&\subseteq\chi_S^c(x)\cup f_S(a)\\&=\phi\cup f_S(a)\\&=f_S(a).\end{aligned}$$

Therefore, (f_S,S) is a uni-soft interior ideal over $U.$

Lemma 2. (see*[31]*).Let $(S,\cdot,\leq)$ be an ordered semigroup. For any non-empty subset A of $S,$ the following conditions are equivalent:

(1) A is a left $(\text{resp.},\text{right})$ ideal of $S.$

(2) The characteristic uni-soft set (χ_A^c,S) over U is a uni-soft left $(\text{resp.},\text{right})$ ideal over U.

4. Uni-soft simple ordered semigroups

In this section, we define uni-soft simple ordered semigroups and characterize simple ordered semigroups in terms of uni-soft interior ideals.

Definition 4. An ordered semigroup S is called uni-soft simple if for any uni-soft ideal of S over U, we have $f_S(x) \subseteq f_S(y)$, for all S $\forall x, y \in S$.

Lemma 3. (see [13]). An ordered semigroup S is left simple if and only if for every $a \in S$, we have $S = (Sa]$.

Theorem 7. Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following conditions are equivalent:

(1) S is a left (resp., right) simple ordered semigroup.

(2) Every uni-soft left (resp., right) ideal (f_S, S) over U is a constant function.

Proof. Let (f_S, S) be a uni-soft left ideal over U and $a, b \in S$. Since S is left simple and $b \in S$, by Lemma 3, we have $S = (Sb]$. Since $a \in S$, we have $a \in (Sb]$, then $a \leq xb$ for some $x \in S$. Since (f_S, S) is a uni-soft left ideal over U , We have

$$f_S(a) \subseteq f_S(xb) \subseteq f_S(b),$$

In a similar way, we can prove that $f_S(b) \subseteq f_S(a)$. Hence $f_S(a) = f_S(b)$. This implies that $f_S : S \to P(U)$ is constant, since a and b are arbitrarily in S. Similarly if S is a right simple ordered semigroup, then every uni-soft right ideal over U is a constant function.

Conversely, assume that (2) holds. Let A be a left ideal of S. Then the characteristic soft set (χ_A^c, S) over U is a uni-soft left ideal over U by lemma 2, and so it is constant by supposition. For any $x \in S$, we have $\chi_A^c(x) = \phi$, since A is non-empty, and thus $x \in A$. This shows that $A = S$. Therefore S is left simple.

In the case that S is right simple, the proof follows similarly.

Theorem 8. Let $(S,\cdot,\leq)$ be a simple ordered semigroup. Then every uni-soft interior ideal over U is constant.

Proof. Suppose that S be a simple ordered semigroup, (f_S, S) be a uni-soft interior ideal over U and $a, b \in S$. Since S is simple and $b \in S$, we have $S = (SbS]$. For $a \in S$ we have $a \in (SbS]$, it follows that $a \leq xby$ for some $x, y \in S$. By hypothesis we have

$$f_S(a) \subseteq f_S(xby) \subseteq f_S(b),$$

Similarly we have $f_S(b) \subseteq f_S(a)$, and so $f_S(a) = f_S(b)$. This shows that $f_S : S \to P(U)$ is constant, since a and b are arbitrarily in S.

Theorem 9. Let $(S,\cdot,\leq)$ be an ordered semigroup. A soft set (f_S, S) over U is a uni-soft interior ideal over U if and only if the non-empty $\delta-$exclusive set of (f_S, S) is an interior ideal of S for all $\delta \in P(U)$.

Proof. Suppose that (f_S, S) is a uni-soft interior ideal over U. Let $\delta \in P(U)$ be such that $e_S(f_S;\delta) \neq \phi$. Let $x, y \in S$ and $a \in e_S(f_S;\delta)$, then $f_S(x) \subseteq \delta$. It follows from hypothesis that

$$f_S(xay) \subseteq f_S(a) \subseteq \delta.$$

Thus $xay \in e_S(f_S;\delta)$. Furthermore, let $x, y \in S$ be such that $x \leq y \in e_S(f_S;\delta)$. Then $f_S(y) \subseteq \delta$. By hypothesis we have

$$f_S(x) \subseteq f_S(y) \subseteq \delta.$$

Which means that $x \in e_S(f_S;\delta)$. Therefore, $e_S(f_S;\delta)$ is an interior ideal of S.

Conversely, assume that the non-empty $\delta-$ exclusive set of (f_S, S) is an interior ideal of S for all $\delta \in P(U)$. Let $x, y, z \in S$ be such that $f_S(y) = \delta$. Then $y \in e_S(f_S;\delta)$. Since $e_S(f_S;\delta)$ is an interior ideal of S, so $xyz \in e_S(f_S;\delta)$. Hence

$$f_S(xyz) \subseteq \delta = f_S(y).$$

Now let $x, y \in S$ with $x \leq y$ be such that $f_S(y) = \delta,$ which means that $y \in e_S(f_S;\delta)$. Since $e_S(f_S;\delta)$ is an interior ideal of S, so $x \in e_S(f_S;\delta)$. Hence $f_S(x) \subseteq \delta = f_S(y)$. Therefore, (f_S, S) is a uni-soft interior ideal over U.

Theorem 10.*Let* $(S,\cdot,\leq)$ *be an ordered semigroup. Then the soft union of two uni-soft interior* $(\text{resp.,left,right})$ *ideals over* U *is also a uni-soft interior* $(\text{resp.,left,right})$ *ideal over* U.

Proof. Let (f_S, S) and (g_S, S) be union soft interior ideals over U. For any $x, a, y \in S$, we have

$$\begin{aligned}(f_S \tilde{\cup} g_S)(xay) &= f_S(xay) \cup g_S(xay)\\ &\subseteq f_S(a) \cup g_S(a)\\ &= (f_S \tilde{\cup} g_S)(a).\end{aligned}$$

Furthermore, let $x, y \in S$ such that $x \leq y$. Since (f_S, S) and (g_S, S) are union soft interior ideals over U, so we have

$$\begin{aligned}(f_S \tilde{\cup} g_S)(x) &= f_S(x) \cup g_S(x)\\ &\subseteq f_S(y) \cup g_S(y)\\ &= (f_S \tilde{\cup} g_S)(y).\end{aligned}$$

Therefore, $(f_S \tilde{\cup} g_S, S)$ is a uni-soft interior ideal over U.

In a similar way, $(f_S \tilde{\cup} g_S, S)$ is a uni-soft left (resp.,right) ideal over U.

Theorem 11.Let $(S,\cdot,\leq)$be an ordered semigroup. Then for any uni-soft set (f_S, S) over U, let (f_S^*, S) be a uni-soft over U defined by

$$f_S^* : S \to P(U),\ x \mapsto \begin{cases} f_S(x) & \text{if } x \in e_S(f_S;\delta), \\ \rho & \text{otherwise,} \end{cases}$$

where δ and ρ are subsets of U with $\bigcup\limits_{x \notin e_S(f_S;\delta)} f_S(x) \not\subset \rho.$ If (f_S, S) is a uni-soft interior ideal over U, then so is (f_S^*, S).

Proof. Let $x, y, z \in S$. If $y \in e_S(f_S;\delta)$, then $xyz \in e_S(f_S;\delta)$, as $e_S(f_S;\delta)$ is an interior ideal of S by Theorem 9. Hence

$$f_S^*(xyz) = f_S(xyz) \subseteq f_S(y) = f_S^*(y).$$

Now if $y \notin e_S(f_S;\delta)$, then $f_S^*(y) = \rho$. Thus $f_S^*(xyz) \subseteq \delta = f_S^*(y)$.

Furthermore, let $x, y \in S$ with $x \leq y$ be such that $f_S^*(y) = \delta$. Which means that $y \notin e_S(f_S;\delta)$. Then either $x \in e_S(f_S;\delta)$ or $x \notin e_S(f_S;\delta)$. In any case

$$f_S^*(x) \subseteq \rho = f_S^*(y).$$

Now if $y \in e_S(f_S;\delta)$. Since $e_S(f_S;\delta)$ is an interior ideal of S by Theorem 9. Then $x \in e_S(f_S;\delta)$. Hence

$$f_S^*(x) = f_S(x) \subseteq f_S(y) = f_S^*(y).$$

Therefore, (f_S^*, S) is a uni-soft interior ideal over U.

Let $(S,\cdot,\leq)$ and $(T,\cdot,\leq)$ be two ordered semigroups. Under the coordinatewise multiplication, i.e.,

$$(x,a)(y,b) = (xy, ab)$$

where $(x,a),(y,b) \in S \times T$, the Cartesian product $S \times T = \{(x,a) | x \in S, a \in T\}$ is a semigroup. Define a partial order $\leq$ on $S \times T$ by

$$(x,a) \leq (y,b) \text{ if and only if } x \leq y \text{ and } a \leq b$$

where $(x,a),(y,b) \in S \times T$. Then, $(S \times T, \cdot, \leq)$ is an ordered semigroup.

For uni-soft sets (f_S, S) and (f_T, T) over U, we consider a uni-soft set $(f_{S \vee T}, S \times T)$ over U in which $f_{S \vee T}$ is defined as follows:

$$f_{S \vee T} : S \times T \to P(U), (x,a) \mapsto f_S(x) \cup f_T(a).$$

Theorem 12. Let $(S, \cdot, \leq)$ be an ordered semigroup. If (f_S, S) and (f_T, T) are uni-soft interior $(\text{resp.}, \text{left}, \text{right})$ ideals over U, then $(f_{S \vee T}, S \times T)$ is a uni-soft interior $(\text{resp.}, \text{left}, \text{right})$ ideal over U.

Proof. Let $(x,a),(y,b),(z,c) \in S \times T$. Then

$$\begin{aligned} f_{S \vee T}((x,a)(y,b)(z,c)) &= f_{S \vee T}(xyz, abc) \\ &= f_S(xyz) \cup f_T(abc) \qquad (1) \end{aligned}$$

Since (f_S, S) and (f_T, T) are uni-soft interior ideal over U. Then $f_S(xyz) \subseteq f_S(y)$ and $f_T(abc) \subseteq f_T(b)$. Hence From equation (1) we have

$$\begin{aligned} f_S(xyz) \cup f_T(abc) &\subseteq f_S(y) \cup f_T(b) \\ &= f_{S \vee T}(y,b). \end{aligned}$$

Furthermore, Let $(x,a),(y,b) \in S \times T$ be such that $(x,a) \leq (y,b)$. Then

$$\begin{aligned} f_{S \vee T}(x,a) &= f_S(x) \cup f_T(a) \\ &\subseteq f_S(y) \cup f_T(b) \\ &= f_{S \vee T}(y,b). \end{aligned}$$

Therefore, $(f_{S \vee T}, S \times T)$ is a uni-soft interior ideal over U.

Similarly, we show that If (f_S, S) and (f_T, T) are uni-soft left $(\text{resp.}, \text{right})$ ideals over U, then $(f_{S \vee T}, S \times T)$ is a uni-soft left $(\text{resp.}, \text{right})$ ideal over U.

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