Classifications of Ordered Semigroups in Terms of Bipolar Fuzzy Bi-Ideals

Tariq Mahmood¹, M. Ibrar², Asghar Khan²*, Hidayat Ullah Khan³ and Fatima Abbas⁴

¹Department of Electronic Engineering, University of Engineering and Technology, Taxila, Sub Campus Chakwal, Pakistan
²Department of Mathematics, Abdul Wali Khan University Mardan, Mardan, Khyber Pakhtunkhwa, Pakistan
³Department of Mathematics, University of Malakand, at Chakdara, District Dir (L), Khyber Pakhtunkhwa Pakistan
⁴Department of Mathematics, Gomal University, D. I. Khan, Khyber Pakhtunkhwa, Pakistan

Received: May 11, 2017
Accepted: July 30, 2017

ABSTRACT

Bipolar fuzzy set is an extension of fuzzy set. In bipolar fuzzy set the range of the membership function is \([-1,1]\), whereas, in fuzzy set it is \([0,1]\). We employed the concept of bipolar fuzzy set theory in the structure of ordered semigroup and introduced a generalization of bipolar fuzzy bi-ideals. This generalized form of bipolar fuzzy bi-ideals is called \((\epsilon, \epsilon \lor q)\)-bipolar fuzzy bi-ideals of ordered semigroups. We obtained some interesting characterization results of ordered semigroup in terms of this new concept.

KEYWORDS: Bipolar fuzzy bi-ideals, bipolar fuzzy point, \((\alpha, \beta)\)-bipolar fuzzy bi-ideals, positive \(t\)-cut, negative \(s\)-cut, \((s,t)\)-cut.

1. INTRODUCTION

The list of abbreviations that are used frequently in this paper are given in see Table 1.

<table>
<thead>
<tr>
<th>Name</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bipolar fuzzy bi-ideals</td>
<td>BFBI</td>
</tr>
<tr>
<td>Bipolar fuzzy set</td>
<td>BFS</td>
</tr>
<tr>
<td>Fuzzy Bi-ideals</td>
<td>FBI</td>
</tr>
<tr>
<td>Generalised Fuzzy Bi-ideals</td>
<td>GFBI</td>
</tr>
<tr>
<td>Ordered Semigroups</td>
<td>OS</td>
</tr>
<tr>
<td>If and only if</td>
<td>iff</td>
</tr>
<tr>
<td>The set of all bipolar fuzzy subsets of (S)</td>
<td>BF(S)</td>
</tr>
<tr>
<td>Ordered Bipolar Fuzzy Point</td>
<td>OBFP</td>
</tr>
<tr>
<td>Bipolar fuzzy bi-ideals</td>
<td>((\alpha, \beta))-BFBI</td>
</tr>
<tr>
<td>((\epsilon, \epsilon \lor q))-bipolar fuzzy generalized bi-ideal</td>
<td>((\epsilon, \epsilon \lor q))-BFGBI</td>
</tr>
<tr>
<td>((\epsilon, \epsilon))-bipolar fuzzy bi-ideal</td>
<td>((\epsilon, \epsilon))-BFBI</td>
</tr>
<tr>
<td>((\epsilon, \epsilon \lor q))-bipolar fuzzy bi-ideal</td>
<td>((\epsilon, \epsilon \lor q))-BFBI</td>
</tr>
<tr>
<td>Bipolar fuzzy left (right) ideal</td>
<td>BFL (R) I</td>
</tr>
<tr>
<td>Bipolar fuzzy ideal</td>
<td>BFI</td>
</tr>
</tbody>
</table>

Zhang (Zhang, 1994, 1998) further generalized the theory of fuzzy set (Zadeh, 1965) and defined BFS. In BFS the range of the membership function is \([-1,1]\) rather than \([0,1]\) (as in case of fuzzy set). In BFS, we split the range \([-1,1]\) into \((0,1)\), \(0\) and \([-1,0)\). If the membership degree of an element \(x\) (say) belongs to the interval \((0,1)\), then it indicates that \(x\) to some extent satisfies the property, whereas, if the membership
degree $x$ is equal to 0, then we say that $x$ is irrelevant to the corresponding property. On the other hand and the membership degree of $x$ belongs to $[-1,0)$, then $x$ satisfies the implicit counter property to a certain degree. Although BFS look like similar to intuitionistic fuzzy sets but they are totally different from each other as mentioned in (Lee, 2004).

Kuroki (Kuroki, 1979, 1981, 1991) was the first to employed the theory of fuzzy set in the structure of semigroups. In (Bhakat and Das, 1992) a generalized form of fuzzy subgroup (Rosenfeld, 1971) is presented. In addition, the generalization of FBI in semigroups is given in (Kazanci and Yamak, 2008). Further, Jun et al (Jun et al., 2005) generalized fuzzy sub-algebra in BCK/BCI algebra and presented the idea of $(\alpha, \beta)$-fuzzy sub-algebra, where $\alpha$ and $\beta$ are in $\{e, q, e \lor q, e \land q\}$ and $\alpha \neq e \land q$. In this regard for further study in different branches of algebra see (Criste, 2010, Davvaz, 2010, Kazanci, 2008, Shabir, 2010, Ma, 2008).

A BFS in a set $S$ is denoted by:

$$f = \{(x, f_s(x) : S \to [-1,0], f_p(x) : S \to [0,1])| x \in S\},$$

where $f_s$ and $f_p$ are called the negative membership and positive membership mappings respectively. If for an element $x$ in $S$ we have $f_p(x) \neq 0, f_s(x) = 0$, then $x$ has a positive satisfaction of a BFS $f$ in $S$. On other hand if $f_s(x) \neq 0, f_p(x) = 0$, then $x$ satisfies the counter property of a BFS $f$ in $S$ up to some extent. There is a possibility for an element $(s) x$ such that $f_p(x) \neq 0 \neq f_s(x)$. For short write $f = (S; f_s, f_p)$ to represent a BFS.

This study aims to introduce a more generalized form of BFBI of OS and investigate several characterization of OS in terms of this new idea. We further discuss relations between $(e, e)$-BFBI, $(e, e \lor q)$-BFBI and a $(q, e \lor q)$-BFBI.

2. PRELIMINARIES

Let $S$ be a non-empty set, “$\cdot$” denotes operation of multiplication and “$\leq$” denotes partial order relation. If $S$ satisfy the following conditions:

(1) $(S, \cdot)$ is a semigroup,
(2) $(S, \leq)$ is a partial ordered set,
(3) $(\forall a, b, x \in S) (a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb$).

Then $(S, \cdot, \leq)$ is called an OS and is denoted by $S$.

If $\phi \neq A, B \subseteq S$ and $x, y, z \in S$, then we denote and define the following:

$$(A) = \{t \in S : t \leq h \in A\}, AB = \{ab : a \in A and b \in B\}, A = \{(y, z) \in S \times S | x \leq yz\},$$

$A \subseteq B \Rightarrow A \subseteq (A),$ $A \subseteq B \Rightarrow \subseteq (A),$ $\subseteq (AB),$ $\subseteq (\{A\}) = (A).$

The characteristic function of $A$ is denoted by $\chi_A = (S, \chi_{\cdot}, \chi_{,})$ where $\chi_{\cdot}$ and $\chi_{,}$ are defined as:

$$\chi_{\cdot}(x) = \begin{cases} -1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

$$\chi_{,}(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

If $A^c \subseteq A$, then $A$ is called a subsemigroup. A subsemigroup $A$ of $S$ is a bi-ideal if:

(1) $ASA \subseteq A$,
(2) $x \leq y \in A \Rightarrow x \in A$,

for all $x, y \in S$.

The negative $s$-cut is defined and denoted as $N(f; s) = \{x \in S | f_s(x) \leq s\}$ whereas the positive $t$-cut is represented by $P(f; t) = \{x \in S | f_p(x) \geq t\}$ for a BFS $f = (S; f_s, f_p)$. We represent and define the $(s, t)$-cut of $f$ as $C(f; (s, t)) = N(f; s) \cap P(f; t)$.
For two BFSs \( f = (S; f_s, f_p) \) and \( g = (S; g_s, g_p) \) and for all \( x \in S \) we say that:

i. \( f \preceq g \) iff \( f_s(x) \geq g_s(x) \) and \( f_p(x) \leq g_p(x) \),

ii. \( f = g \) iff \( f \preceq g \) and \( g \preceq f \),

iii. \( f \wedge g = (S; f_s \land g_s, f_p \land g_p) \),

iv. \( f \lor g = (S; f_s \lor g_s, f_p \lor g_p) \),

For all \( x \in S \), the BFS \( 0 = (S; 0_s, 0_p) \) and \( 1 = (S; 1_s, 1_p) \) are defined as follows:

\( 0_s(x) = 0 = 0_p(x) \) and \( 1_s(x) = 1 \) and \( 1_p(x) = 1 \).

The product of \( f = (S; f_s, f_p) \) and \( g = (S; g_s, g_p) \) is defined and denoted as:

\[
(f \circ g)(x) = \begin{cases} 
\land \{f_s(y) \lor g_s(z)\} & \text{if } A_s \neq \phi, \\
0 & \text{if } A_s = \phi.
\end{cases}
\]

and

\[
(f \circ g)(x) = \begin{cases} 
\lor \{f_s(y) \land g_s(z)\} & \text{if } A_p \neq \phi, \\
0 & \text{if } A_p = \phi.
\end{cases}
\]

The multiplication “\( \circ \)” is well defined and associative and clearly \( (BF(S), \circ, \preceq) \) is an ordered semigroup.

**Definition 2.1** (Shabir, 2013): A BFS \( f = (S; f_s, f_p) \) in \( S \) is called a BFL (R) I of \( S \) if:

1) \( f_s(x) \leq f_p(x) \) and \( f_p(x) \geq f_s(y) \)

2) \( f_s(xy) \leq f_s(y) \) and \( f_p(xy) \geq f_p(y) \)

\( (f_s(xy) \leq f_s(x) \) and \( f_p(xy) \geq f_p(x) \))

for all \( x, y \in S \).

A BFS is called BFI of \( S \) if it is both BFLI and BFRI.

**Definition 2.2** (Shabir, 2013): A BFS \( f \) is called BFBI of \( S \) if:

1) \( f_s(x) = f_s(y) \) and \( f_p(x) \geq f_p(y) \)

2) \( f_s(xy) = f_s(y) \) and \( f_p(xy) \geq f_p(y) \)

\( (f_s(xy) = f_s(x) \) and \( f_p(xy) \geq f_p(x) \))

for all \( x, y, z \in S \).

**Theorem 2.3** Let \( f \) be a BFS in \( S \). Then \( f \) is a BFBI iff \( C(f; (s, t)) \neq \emptyset \) is a bi-ideal of \( S \) for all \( (s, t) \in [0, 1] \times [0, 1] \).

**Proposition 2.4** The following hold for \( A, B \subseteq S \).

i) \( A \subseteq B \) if and only if \( X_a \subseteq X_b \).

ii) \( X_a \cap X_b = X_{a \lor b} \).

iii) \( X_a \circ X_b = X_{a \land b} \).

Proof. (i) It is straightforward to prove and thus omitted.

(ii) First we consider \( A \cap B = \emptyset \). Then clearly \( X_a \cap X_b = X_{a \lor b} \). On the other hand if \( A \cap B \neq \emptyset \), then we let \( x \in A \cap B \) and therefore

\[
X_{a \lor b}(x) = -1 = (X_a \land X_b)(x),
\]

and

\[
X_{a \land b}(x) = 1 = (X_a \land X_b)(x).
\]

Hence \( X_a \lor X_b = X_{a \lor b} \).

(iii) Now we prove \( X_a \circ X_b = X_{a \land b} \). For this it is enough to show that \( X_{a \lor b} \circ X_{a \land b} = X_{a \land b} \) and \( X_{a \land b} \circ X_{a \lor b} = X_{a \land b} \). Let \( x \in S \): If \( x \in (AB) \), then \( X_{a \land b}(x) = 1 \) and \( X_{a \lor b}(x) = -1 \) for some \( a \in A \) and \( b \in B \). Hence \( (a, b) \in A \) and therefore \( A \neq \emptyset \) and we have
\((\chi \circ \chi_\beta)(x) = \bigwedge_{(y,z) = a, b} \{ \chi_\alpha(y) \lor \chi_\beta(z) \}\)
\[\leq \{ \chi_\alpha(a) \lor \chi_\beta(b) \} = -1,\]

and
\[(\chi \circ \chi_\beta)(x) = \bigvee_{(y,z) = a, b} \{ \chi_\alpha(y) \land \chi_\beta(z) \}\]
\[\geq \{ \chi_\alpha(a) \land \chi_\beta(b) \} = 1.\]

It follows that \((\chi \circ \chi_\beta)(x) = -1 = (\chi \circ \chi_\beta)(x)\) and \((\chi \circ \chi_\beta)(x) = 1 = (\chi \circ \chi_\beta)(x)\).

3. Generalized BFBI of OS

The idea of BFBI is generalized to introduce the notion of \((\alpha, \beta)\)-BFBI in OS in this section.

If \((s, t) \in [-1, 0] \times (0, 1]\), then a BFS \(f = (S; f_s, f_p)\) of the form:
\[
f_s(y) = \begin{cases} 
1, & \text{if } y \in (x], \\
0, & \text{if } y \notin (x],
\end{cases}
\]
\[
f_p(y) = \begin{cases} 
1, & \text{if } y \in (x], \\
0, & \text{if } y \notin (x].
\end{cases}
\]

is called an OBFP with \((s, t)\) are the value and support of the OBFP and is denoted by \(\frac{\chi_\alpha \chi_\beta}{s, t}\). An OBFP \(\frac{\chi_\alpha \chi_\beta}{s, t}\) is said to belongs to \(f\) (denoted as \(\frac{\chi_\alpha \chi_\beta}{s, t} \in f\)) if \(f_s(x) \leq s\) and \(f_p(x) \geq t\). If \(f_s(x) + s < -1\) and \(f_p(x) + t > 1\), then \(\frac{\chi_\alpha \chi_\beta}{s, t}\) is said to quasi-coincidence with \(f\) (denoted as \(\frac{\chi_\alpha \chi_\beta}{s, t} \in qf\)). If \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in f\) or \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in qf\), then we write \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in qf\). By \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in qf\) we mean \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in f\) and \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in qf\). We write \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in qf\) if the relation \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in qf\) does not hold.

Let \(f_s(x) \geq -0.5\) and \(f_p(x) \leq 0.5\) for all \(x \in S\) \(s, t) \in [-1, 0] \times (0, 1]\) such that \(\frac{\chi_\alpha \chi_\beta}{s, t} = qf\). Then \(f_s(x) \leq s\), \(f_p(x) \geq t\), \(f_s(x) + s < -1\), and \(f_p(x) + t > 1\). It follows that \(\frac{\chi_\alpha \chi_\beta}{s, t} \not\in qf\). From the above discussion we conclude that the case \(\alpha \not\in qf\) will be omitted in this study.

**Definition 3.1** A BFS \(f\) in \(S\) is called \((\alpha, \beta)\)-bipolar fuzzy left (right) ideal of \(S\) where \(\alpha \not\in qf\) if the following conditions hold for all \((s, t) \in [-1, 0] \times (0, 1]\) and \(x, y \in S\):

1. If \(x \leq y\), then \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\).
2. \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\) \(\Rightarrow \frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\).

**Definition 3.2** A BFS \(f\) in \(S\) is called \((\alpha, \beta)\)-BFBI the following conditions hold for all \((s, t) \in [-1, 0] \times (0, 1]\) and \(x, y, z \in S\):

1. If \(x \leq y, z \in S\), then \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\).
2. \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\) \(\Rightarrow \frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\).
3. \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\) \(\Rightarrow \frac{\chi_\alpha \chi_\beta}{x, y} \not\in qf\).

**Theorem 3.3** Let \(f\) be a BFS in \(S\). Then \(f\) is a BFBI if and only if for all \((s, t) \in [-1, 0] \times (0, 1]\) and \(x, y, z \in S\) the following conditions hold simultaneously:

1. If \(x \leq y\), then \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\).
2. \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\) \(\Rightarrow \frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\).
3. \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\) \(\Rightarrow \frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\).

**Proof.** Suppose that \(f\) is a BFBI. If \(x, y \in S\) such that \(x \leq y\) and \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\), then \(f_s(x) \leq s\) and \(f_p(y) \geq t\). By Definition 2.2 we have \(f_s(x) \leq f_s(y)\) and \(f_p(y) \geq f_p(y)\). This implies that \(f_s(x) \leq s\) and \(f_p(x) \geq t\) and so \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\).

If \(\frac{\chi_\alpha \chi_\beta}{x, y} \not\in f\), then \(f_s(x) \leq s\), \(f_p(y) \geq t\), and \(f_s(y) \leq s\), \(f_p(y) \geq t\). By Definition 2.2 we
have
\[ f_s(xy) \leq \lor \{ f_s(x), f_s(y) \} \leq \lor \{ s_t, s_z \} \]
and
\[ f_p(xy) \geq \land \{ f_p(x), f_p(y) \} \geq \land \{ t_t, t_z \} , \]
it follows that \( \frac{x}{y} \in f \).
If \( x, y, z \in S \) such that \( \frac{x}{y} \in f \) and \( \frac{y}{z} \in f \), then \( f_s(x) \leq s_t, f_s(x) \geq t_t \) and \( f_s(z) \leq s_z, f_s(z) \geq t_z \). By Definition 2.2 (3) we have
\[ f_s(xy) \leq \lor \{ f_s(x), f_s(y) \} \leq \lor \{ s_t, s_z \} \]
and
\[ f_p(xy) \geq \land \{ f_p(x), f_p(y) \} \geq \land \{ t_t, t_z \} , \]
this implies \( \frac{x}{y} \in f \).
Conversely let \( x, y, z \in S \) such that \( x \leq y \). If \( f_s(y) = s \) and \( f_p(y) = t \), then \( \frac{s}{t} \in f \) and hence by (1) we have \( f_s(x) \leq s_t, f_s(x) \geq t_t \). This implies \( f_s(x) \leq s = f_s(y) \) and \( f_s(x) \geq t = f_p(y) \). Hence we have \( f_s(x) \leq f_s(y) \) and \( f_p(x) \geq f_p(y) \).
If \( \frac{x}{y} \in f \) and \( \frac{y}{z} \in f \), then by (2) we have \( \frac{y}{z} \in f \). This implies that \( f_s(xy) \leq \lor \{ f_s(x), f_s(y) \} \) and \( f_p(xy) \geq \land \{ f_p(x), f_p(y) \} \).
Let \( x, y, z \in S \) such that \( \frac{x}{y} \in f \) and \( \frac{y}{z} \in f \), then by (3) we have \( \frac{y}{z} \in f \). It follows that
\[ f_s(xy) \leq \lor \{ f_s(x), f_s(y) \} \quad \text{and} \quad f_p(xy) \geq \land \{ f_p(x), f_p(y) \} . \]

**Proposition 3.4** (Ibrar et al., 2016) A BFS \( f \) in \( S \) is an \( (e, e \lor q) \)-BFGBI iff the following conditions hold simultaneously for all \( x, y, z \in S \):

1. \( x \leq y \Rightarrow f_s(x) \leq \lor \{ f_s(y), -0.5 \} \) and \( f_p(x) \geq \land \{ f_p(y), 0.5 \} \),
2. \( f_s(xy) \leq \lor \{ f_s(x), f_s(z) \} \) and \( f_p(xy) \geq \land \{ f_p(x), f_p(z) \} \).

**Theorem 3.5** A BFS \( f \) in \( S \) is an \( (e, e \lor q) \)-BFBI iff the following conditions hold simultaneously for all \( x, y, z \in S \):

1. \( x \leq y \Rightarrow f_s(x) \leq \lor \{ f_s(y), -0.5 \} \) and \( f_p(x) \geq \land \{ f_p(y), 0.5 \} \),
2. \( f_s(xy) \leq \lor \{ f_s(x), f_s(y), -0.5 \} \) and \( f_p(xy) \geq \land \{ f_p(x), f_p(y), 0.5 \} \),
3. \( f_s(xy) \leq \lor \{ f_s(x), f_s(z), -0.5 \} \) and \( f_p(xy) \geq \land \{ f_p(x), f_p(z), 0.5 \} \).

**Proof.** The proof follows from Proposition 3.4.

**Theorem 3.6** Every \( (e, e \lor q) \)-BFBI is \( (e, e \lor q) \)-BFBI.

**Proof.** It can be proved easily.

The below given example shows that every \( (e, e \lor q) \)-BFBI is not \( (e, e) \)-BFBI.

**Example 3.7** Let \( S = \{a, b, c, d, e\} \) set be a be set with the following multiplication table and order relation “\( \leq \)”

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>d</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>d</td>
<td>e</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>d</td>
<td>a</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>d</td>
<td>a</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>e</td>
<td>a</td>
<td>d</td>
<td>c</td>
<td>d</td>
<td>e</td>
</tr>
</tbody>
</table>

\( \leq \) \( \{(a, a), (a, c), (a, d), (a, e), (b, b), (c, c), (c, e), (d, d), (e, e)\} \).

Then \( S \) is an OS. Let \( f \) be a BFS in \( S \) defined by Table 2 below:
Then $f$ is $(\epsilon,\epsilon \lor q)$-BFBI but not an $(\epsilon,\epsilon)$-BFBI because $\frac{a}{\epsilon}$ and $\frac{a}{\epsilon}$ are not in $f$. Let $s, t, x, y \in S$ such that $x \leq y$ and $(s, t) = [0,1] \times (0,1]$. If $f_s(t) \in f$ then $f_s(t) \in \epsilon q f$ and hence by hypothesis we have $f_s(t) \in \epsilon q f$. Next we consider $f_s(t) \in f$, $f_s(t) \in \epsilon q f$ and hence by hypothesis this implies $f(t) \in \epsilon q f$. Again we let $x, y, z \in S$ such that $f(t) \in f$ and $f(t) \in f$ implies $f(t) \in \epsilon q f$. By hypothesis $f(t) \in \epsilon q f$. Hence $f = (S; f_s(t), f)$ is an $(\epsilon,\epsilon \lor q)$-BFBI.

**Theorem 3.9** If $f$ is a nonzero $(\epsilon,\epsilon \lor q)$-BFBI of $S$, then the set $S = \{x \in S \mid f_s(x) \neq 0 \} \cap \{x \in S \mid f_s(x) \neq 0 \}$ is a bi-ideal of $S$.

**Proof.** Let $x, y \in S$ such that $x \leq y$ and $y \in S$. Then $f_s(y) \neq 0$ and $f_s(y) \neq 0$. Hence $f_s(y) = 0$ and $f_s(y) > 0$. Suppose that $f_s(x) = 0$ or $f_s(y) = 0$. Since $f_s(x) = 0$ or $f_s(y) = 0$, this implies that $f_s(x) \in f$ and $f_s(y) = f_s(y)$. Then $f_s(x) \in f$, which is a contradiction. Also $f_s(x) + f_s(y) = f_s(y) = 0$, $f_s(x) + f_s(y) = 0$, $f_s(x) + f_s(y) = 0$. This implies that $f_s(x, y) = 0$, which is a contradiction. Therefore, $f_s(x) = 0$ and $f_s(y) = 0$. Hence, $x \in S$.

If $x, y \in S$, then $f_s(x) \neq 0$, $f_s(y) \neq 0$, $f_s(x) \neq 0$ and $f_s(y) \neq 0$. So $f_s(x) = 0$, $f_s(y) = 0$, $f_s(x) > 0$, and $f_s(y) > 0$. Suppose that $x y \in S$, then $f_s(x y) = 0$ or $f_s(x y) = 0$. Clearly $f_s(x y) = 0$ and $f_s(x y) = 0$. Since $f_s(x y) = 0$ or $f_s(x y) = 0$, this implies that $f_s(x y) \neq 0$. Also $f_s(x y) = 0$, $f_s(x y) = 0$, $f_s(x y) = 0$. This implies that $f_s(x y) \neq 0$, which is a contradiction. Therefore, $f_s(x y) \neq 0$ and $f_s(x y) \neq 0$. Hence, $x y \in S$.

Let $x, y, z \in S$ such that $x, z \in S$. Then $f_s(x) \neq 0$, $f_s(z) \neq 0$, $f_s(x) \neq 0$ and $f_s(z) \neq 0$. So $f_s(x) < 0$, $f_s(z) > 0$, $f_s(x) > 0$ and $f_s(z) > 0$. If $x y z \subseteq S$, then $f_s(x y z) = 0$ or $f_s(x y z) = 0$. Clearly $f_s(x y z) = 0$ and $f_s(x y z) = 0$. Since $f_s(x y z) = 0$, $f_s(x y z) = 0$, $f_s(x y z) = 0$. This implies that $f_s(x y z) \neq 0$, which is a contradiction. Also $f_s(x y z) + f_s(x) \lor f_s(z) = f_s(x) \lor f_s(z) \neq 0$ or $f_s(x y z) + f_s(x) \lor f_s(z) = f_s(x) \lor f_s(z) \neq 0$. This implies that $f_s(x y z) \neq 0$, which is a contradiction. Therefore, $x y z \in S$.

**Theorem 3.10** Let $f$ be an $(\epsilon,\epsilon \lor q)$-BFBI of $S$ and $(f_s(x), f_s(x)) = (-0.5,0) \times (0,0.5)$ for all $x \in S$. Then $f$ is an $(\epsilon,\epsilon)$-BFBI.

**Proof.** Suppose $f$ be an $(\epsilon,\epsilon \lor q)$-BFBI of $S$.

Let $x, y \in S$ be such that $x \leq y$ and $x \in S$, then $f_s(y) \leq s$ and $f_s(y) \geq t$. By

<table>
<thead>
<tr>
<th>$S$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_s$</td>
<td>-0.75</td>
<td>-0.35</td>
<td>-0.72</td>
<td>-0.58</td>
<td>-0.65</td>
</tr>
<tr>
<td>$f_s$</td>
<td>0.8</td>
<td>0.3</td>
<td>0.7</td>
<td>0.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 2
Theorem 3.5 (1) we have \( f_s(x) \leq f_s(y) + 0.5 \leq s \) and \( f_p(x) \geq f_p(y) \wedge 0.5 \geq t \) and so \( \frac{x}{x+y} \in f \).

Let \( x, y \in S \) and \( (s_1, t_1), (s_2, t_2) \in [-1,0] \times (0,1] \) be such that \( \frac{x}{x+y}, \frac{y}{y+t} \in f \) then \( f_s(x) \leq s_1, f_s(y) \leq s_2 \) and \( f_p(x) \geq t_1, f_p(y) \geq t_2 \). By Theorem 3.5 (2) we have \( f_s(xy) \leq \vee \{ f_s(x), f_s(y), -0.5 \} \leq \vee \{ s_1, s_2 \} \) and \( f_p(xy) \geq \wedge \{ f_p(x), f_p(y), 0.5 \} \geq \wedge \{ t_1, t_2 \} \), which implies \( \frac{xy}{(\vee \{ s_1, s_2 \}, \wedge \{ t_1, t_2 \})} \in f \).

Theorem 3.11 Let \( I \) be a bi-ideals of \( S \) and \( f \) a BFS in \( S \) such that

(i) \( f_s(x) = 0 = f_p(x) \) for all \( x \in S \setminus I \),
(ii) \( (f_s(x), f_p(x)) \in [-1,0] \times [0,1] \) for all \( x \in I \).

Then \( f \) is \( (q, \in \vee q) \)-BFBI of \( S \).

Proof. Let \( x, y \in S \) such that \( x \leq y \) and \( (s, t) \in [-1,0] \times (0,1] \). Let \( \frac{x}{x+y}qf \) then \( f_s(y) + s < -1 \) and \( f_p(y) + t > 1 \). This implies that \( y \in I \). Since \( I \) is bi-ideals of \( S \), so \( x \in I \). In order to check \( \frac{x}{x+y} \in \vee qf \), we consider the following four cases:

1. \( s \geq 0.5 \) and \( t \leq 0.5 \),
2. \( s < 0.5 \) and \( t > 0.5 \),
3. \( s < 0.5 \) and \( t \leq 0.5 \),
4. \( s \geq 0.5 \) and \( t > 0.5 \).

The first case induces \( f_s(x) \leq -0.5 \leq s \) and \( f_p(x) \geq 0.5 \geq t \) and hence \( \frac{x}{x+y} \in f \). The second case implies that \( f_s(x) + s < -1 \) and \( f_p(x) + t > 1 \), it follows that \( \frac{x}{x+y}qf \). Since \( \frac{x}{x+y}qf \) so case (3) and (4) do not occur. Consequently, \( \frac{x}{x+y} \in \vee qf \).

Let \( x, y \in S \) such that \( \frac{x}{x+y}qf \) and \( \frac{y}{y+t}qf \) for \( (s_1, t_1), (s_2, t_2) \in [-1,0] \times (0,1] \), then \( f_s(x) + s_1 < -1 \), \( f_s(y) + s_2 < -1 \), \( f_p(x) + t_1 > 1 \) and \( f_p(y) + t_2 > 1 \) and so \( x, y \in I \). Since \( I \) is bi-ideals of \( S \), therefore \( xy \in I \). In order to check \( \frac{xy}{x+y}qf \) we consider the following four cases:

1. \( s_1 \vee s_2 \geq -0.5 \) and \( t_1 \wedge t_2 \leq 0.5 \),
2. \( s_1 \vee s_2 \leq -0.5 \) and \( t_1 \wedge t_2 > 0.5 \),
3. \( s_1 \vee s_2 \leq -0.5 \) and \( t_1 \wedge t_2 \leq 0.5 \),
4. \( s_1 \vee s_2 \geq -0.5 \) and \( t_1 \wedge t_2 > 0.5 \).

The first case induces \( f_s(xy) \leq -0.5 \leq s_1 \wedge s_2 \) and \( f_p(xy) \geq 0.5 \geq t_1 \wedge t_2 \) and so \( \frac{xy}{x+y} \in f \). The second case implies that \( f_s(xy) + s_1 \wedge s_2 < -1 \) and \( f_p(xy) + t_1 \wedge t_2 > 1 \) that is \( \frac{xy}{x+y}qf \). Since \( \frac{xy}{x+y}qf \) and \( \frac{x}{x+y}qf \) so cases (3) and (4) do not occur. Consequently, \( \frac{xy}{x+y} \in \vee qf \).

Let \( x, y, z \in S \) such that \( \frac{x}{x+y}qf \) and \( \frac{y}{y+t}qf \) for \( (s_1, t_1), (s_2, t_2) \in [-1,0] \times (0,1] \), then \( f_s(x) + s_1 < -1 \), \( f_s(z) + s_2 < -1 \), \( f_p(x) + t_1 > 1 \) and \( f_p(z) + t_2 > 1 \), in which it follows that \( x, z \in I \). Since \( I \) is bi-ideals of \( S \), therefore we have \( xyz \in I \). Now to show that \( \frac{xyz}{x+y+z} \in \vee qf \) we consider the following four cases:

1. \( s_1 \vee s_2 \geq -0.5 \) and \( t_1 \wedge t_2 \leq 0.5 \),
2. \( s_1 \vee s_2 \leq -0.5 \) and \( t_1 \wedge t_2 > 0.5 \),
3. \( s_1 \vee s_2 \leq -0.5 \) and \( t_1 \wedge t_2 \leq 0.5 \),
4. \( s_1 \vee s_2 \geq -0.5 \) and \( t_1 \wedge t_2 > 0.5 \).
The first case induces \( f_p(xy) \leq 0.5 \leq s \lor s \) and \( f_p(xy) \geq 0.5 \geq t \land t \) and so \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in f \). The second case implies that \( f_p(xy) + s \lor s < -1 \) and \( f_p(xy) + t \land t > 1 \) and therefore \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in f \). Since \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \) and \( \frac{n_{t s} - BFBI}{n_{s t} - BFBI} \) and therefore both Case (3) and Case (4) do not occur. Hence \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in \lor qf \).

In general every \( (\epsilon, \epsilon \lor q) - BFBI \) may not be \( (q, q \lor q) - BFBI \) as shown in the following example.

**Example 3.12** Consider the OS of Example 3.7 and define BFS \( f \) in \( S \) as in Table 3.

**Table 3**

<table>
<thead>
<tr>
<th>( S )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
<th>( e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_p )</td>
<td>-0.9</td>
<td>-0.35</td>
<td>-0.72</td>
<td>-0.58</td>
<td>-0.65</td>
</tr>
<tr>
<td>( f_q )</td>
<td>0.8</td>
<td>0.3</td>
<td>0.7</td>
<td>0.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Then \( f \) is an \( (\epsilon, \epsilon \lor q) - BFBI \) of \( S \) but not a \( (q, q \lor q) - BFBI \) because \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \) and \( \frac{n_{t s} - BFBI}{n_{s t} - BFBI} \) but \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in \lor qf \).

In the following theorem we provide a condition for an \( (\epsilon, \epsilon \lor q) - BFBI \) to be \( (q, q \lor q) - BFBI \).

**Theorem 3.13** Every \( (\epsilon, \epsilon \lor q) - BFBI \) is an \( (q, q \lor q) - BFBI \) of \( S \) for condition that \( (s, t) \in [0, 0.5) \times (0, 0.5] \).

**Proof.** Let \( f \) be an \( (\epsilon, \epsilon \lor q) - BFBI \) of \( S \) and \( x, y \in S \) such that \( x \leq y \) and \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \) for \( (s, t) \in [0, 0.5) \times (0, 0.5] \). Then \( f_p(x) + s < 1 \) and \( f_p(y) + t > 1 \) that is \( f_p(x) < 1 - s \leq s \) and \( f_p(y) > 1 - t \geq t \). Hence \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in f \) and \( \frac{n_{t s} - BFBI}{n_{s t} - BFBI} \in f \) and since \( f \) is an \( (\epsilon, \epsilon \lor q) - BFBI \) of \( S \), Therefore \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in \lor qf \).

Let \( x, y \in S \) such that \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \) for some \( (s, t), (s, t) \in [-0.5, 0.5) \times (0, 0.5] \), then \( f_p(x) + s < 1 \), \( f_p(x) + s < 1 \), \( f_p(x) + t > 1 \) and \( f_p(y) + t > 1 \). This implies that \( f_p(x) < 1 - s \leq s \), \( f_p(y) < 1 - t \leq t \), \( f_p(x) > 1 - t \geq t \), and \( f_p(y) > 1 - t \geq t \). Hence \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in f \) and \( \frac{n_{t s} - BFBI}{n_{s t} - BFBI} \in f \) and since \( f \) is an \( (\epsilon, \epsilon \lor q) - BFBI \) of \( S \), therefore \( \frac{n_{s \lor s} - BFBI}{n_{t \land t} - BFBI} \in \lor qf \) and hence \( f \) is \( (q, q \lor q) - BFBI \) of \( S \).

**REFERENCES**


