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The Optimal Homotopy Asymptotic Method with Application to Homogeneous Nonlinear Advection Equations

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ABSTRACT

In this article Optimal Homotopy Asymptotic Method (OHAM) is used for the solution of homogenous advection equations. The accuracy of the method is analyzed by its comparison with exact and Homotopy Perturbation transforms Method (HPTM) solution. The absolute errors and order of approximation presented.
KEYWORDS: OHAM, nonlinear homogenous advection equations.

1. INTRODUCTION

The nonlinear phenomena play a vital role in the field of applied mathematics, physics and engineering. Like the advection problem of the form

$$\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial u(x,t)}{\partial x}=h(x,t), \quad u(x,0)=g(x). \tag{1.1}$$

arise in physical, engineering and applied sciences. A limited number of analytic techniques were used for its solution [1]. Therefore, the researchers paid special attention to search new analytic techniques for the solution of nonlinear partial differential equations. Most of the techniques like Adomian decomposition method (ADM) [2], Variational iterative method (VIM) [3], Differential transform method (DTM) [4], Weighted finite difference method [5], The sinh-cosh method [6], and Homotopy perturbation method (HPM) [7], were used for the solution of weakly nonlinear partial differential equations. For the solution of the strongly nonlinear problems the perturbation methods were studied [8,9,10]. These methods include a small parameter which cannot be found easily. To handle these problem new analytic methods such as artificial parameters method [11], Homotopy analysis method (HAM) [12] and Homotopy perturbation method (HPM) [7] were introduced. These methods pooled the homotopy with the perturbation techniques. Recently, Vasile Marinca *et al* introduced OHAM [13-15] for the solution of nonlinear problems which made the perturbation methods independent of the assumption of small parameters and huge computational work.

The motivation of this paper is to boost OHAM for the solution of nonlinear homogenous advection equations. Ullah et.al. have extended and applied OHAM to different problems [16-31]. In this paper, we have proved that OHAM is useful and reliable for the solution of advection equations, showing its validity and great potential for the solution of transient physical phenomenon in science and engineering.

In the succeeding section, the basic idea of OHAM [6-10] is formulated for the solution of partial differential equations. In Section 3, the effectiveness of the enhanced formulation of OHAM for time-dependant problems has been studied.

2. Application of OHAM to homogenous advection Equations

Model. [32] Here we consider homogenous advection equation with the initial condition

$$\frac{\partial \zeta}{\partial t}+\zeta\frac{\partial \zeta}{\partial x}=0, \ \zeta=-x, \tag{2.1}$$

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The exact solution is

$$\zeta = \frac{x}{t-1}. \tag{2.2}$$

Zeroth Order Problem

$$\frac{\partial \zeta_0}{\partial t} = 0, \zeta_0(x,0) = -x \tag{2.3}$$

Its solution is

$$\zeta_0(x) = -x \tag{2.4}$$

First Order Problem

$$\frac{\partial \zeta_1}{\partial t} = (1+C_1)\frac{\partial \zeta_0}{\partial t} + C_1\zeta_0\frac{\partial \zeta_0}{\partial x}, \zeta_1(x,0) = 0, \tag{2.5}$$

Its solution is

$$\zeta_1 = txC_1. \tag{2.6}$$

Second Order Problem

$$\frac{\partial \zeta_2}{\partial t} = \begin{Bmatrix} (1+C_1)\frac{\partial \zeta_1}{\partial t} + C_2(1+\zeta_0)\frac{\partial \zeta_0}{\partial t} \\ +C_1\zeta_0\frac{\partial \zeta_1}{\partial x} + C_1\zeta_1\frac{\partial \zeta_0}{\partial x} \end{Bmatrix}, \zeta_2(x,0) = 0, \tag{2.7}$$

Its solution is

$$\zeta_2 = C_1xt + C_1^2xt - C_1^2xt^2 + C_2xt \quad . \tag{2.8}$$

Third Order Problem

$$\frac{\partial \zeta_3}{\partial t} = \begin{bmatrix} C_3\frac{\partial \zeta_0}{\partial t} + C_2\frac{\partial \zeta_1}{\partial t} + (1+C_1)\frac{\partial \zeta_2}{\partial t} \\ +C_2\zeta_1\frac{\partial \zeta_0}{\partial t} + C_2\zeta_2\frac{\partial \zeta_0}{\partial x} \\ +(C_1\zeta_1 + C_2\zeta_0)\frac{\partial \zeta_1}{\partial x} + C_1\zeta_0\frac{\partial \zeta_2}{\partial x} \end{bmatrix}, \zeta_3(x,0) = 0, \tag{2.9}$$

Whose solution is

$$\zeta_3 = \begin{bmatrix} C_1xt + 2C_1^2xt - 2C_1^2xt^2 + C_1^3xt - 2C_1^3xt^2 + C_1^3xt^3 \\ +C_2xt + 2C_1C_2xt - 2C_1C_2xt^2 + C_3xt \end{bmatrix}. \tag{2.10}$$

Adding Eqs. (3.1.9), (3.1.12), (3.1.15) and (3.1.18), we obtain:

$$\zeta = \begin{bmatrix} -x + 3C_1xt + 3C_1^2xt - 3C_1^2xt^2 + C_1^3xt - 2C_1^3xt^2 + C_1^3xt^3 + 2C_2xt \\ +2C_1C_2xt - 2C_1C_2xt^2 + C_3xt \end{bmatrix}. \tag{2.11}$$

For $C_1 = -1.0717403328569475$, $C_2 = 0.0002568456183225209$, $C_3 = 0.00003351161458234207$.

$$\zeta = -\begin{bmatrix} x + 1.00037256496904xt + 0.9832709819011352xt^2 \\ +1.2310302488494396xt^3 \end{bmatrix}. \tag{2.12}$$

Solution of the problem by Homotopy Perturbation Transform Method (HPTM)

$$\zeta(x,t) = -x\left(1+t+t^2+t^3+t^4\right). \tag{2.13}$$

Table. 1. Comparison of results

x	OHAM	Exact	HPTM
0.1	-0.11111	-0.111111	-0.11111
0.2	-0.22222	-0.222222	-0.22222
0.3	-0.33333	-0.333333	-0.33333
0.4	-0.44444	-0.444444	-0.44444
0.5	-0.55555	-0.555556	-0.55555
0.6	-0.666661	-0.666667	-0.66666
0.7	-0.777771	-0.777778	-0.77777
0.8	-0.888881	-0.888889	-0.88888
0.9	-0.999991	-1	-0.99999
1.0	-1.1111	-1.11111	-1.11111

Table 2: Absolute errors

$x \therefore t$	$t = 0.01$	$t = 0.015$	$t = 0.1$	$t = 0.15$	$t = 0.2$
0.1	2.73368×10^{-7}	2.55278×10^{-7}	1.01145×10^{-6}	1.36379×10^{-5}	7.46406×10^{-5}
0.2	4.54735×10^{-7}	5.10555×10^{-7}	2.02291×10^{-6}	2.72759×10^{-5}	1.49281×10^{-4}
0.3	6.82103×10^{-7}	7.65833×10^{-7}	3.03436×10^{-6}	4.09138×10^{-5}	2.23922×10^{-4}
0.4	9.09471×10^{-7}	1.02111×10^{-6}	4.04582×10^{-6}	5.45517×10^{-5}	2.98562×10^{-4}
0.5	1.13684×10^{-6}	1.27639×10^{-6}	5.05727×10^{-6}	6.81897×10^{-5}	3.73203×10^{-4}
0.6	1.36421×10^{-6}	1.53167×10^{-6}	6.06873×10^{-6}	8.18276×10^{-5}	4.47843×10^{-4}
0.7	1.59157×10^{-6}	1.78694×10^{-6}	7.08018×10^{-6}	9.54655×10^{-5}	5.22484×10^{-4}
0.8	1.81894×10^{-6}	2.04222×10^{-6}	8.09164×10^{-6}	1.09103×10^{-4}	5.97125×10^{-4}
0.9	2.04631×10^{-6}	2.29450×10^{-6}	9.10309×10^{-6}	1.22741×10^{-4}	6.71765×10^{-4}
1.0	2.27368×10^{-5}	2.55278×10^{-6}	1.01145×10^{-5}	1.36379×10^{-4}	7.46406×10^{-4}

Table 3: Comparison of zeroth order, first order and second order and third order absolute error corresponding to at time $t = 0.1$, and $0 \le x \le 1$

x	Zeroth Order Absolute Error	First Order Absolute Error	Second Order Absolute Error	Third Order Absolute Error
0.1	1.111111×10^{-2}	3.93708×10^{-4}	1.65190×10^{-5}	1.01145×10^{-6}
0.2	2.222222×10^{-2}	7.87416×10^{-4}	3.30380×10^{-5}	2.02291×10^{-6}
0.3	3.333333×10^{-2}	1.18112×10^{-3}	4.95569×10^{-5}	3.03436×10^{-6}
0.4	4.444444×10^{-2}	1.57483×10^{-3}	6.60759×10^{-5}	4.04582×10^{-6}
0.5	5.555556×10^{-2}	1.96854×10^{-3}	8.25949×10^{-5}	5.05727×10^{-6}
0.6	6.666667×10^{-2}	2.36225×10^{-3}	9.91139×10^{-5}	6.06873×10^{-6}
0.7	7.777778×10^{-3}	2.75595×10^{-3}	1.15633×10^{-4}	7.08018×10^{-6}
0.8	8.888889×10^{-3}	3.14966×10^{-3}	1.32152×10^{-4}	8.09164×10^{-6}
0.9	1.000000×10^{-1}	3.54337×10^{-3}	1.48671×10^{-4}	9.10309×10^{-6}
1.0	1.111111×10^{-1}	3.93708×10^{-3}	1.65190×10^{-4}	1.01145×10^{-5}

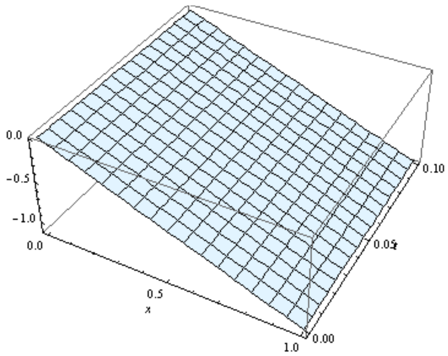

Fig 1, 3D Approximate solution of $\zeta(x,t)$ for $t = 0.5$.

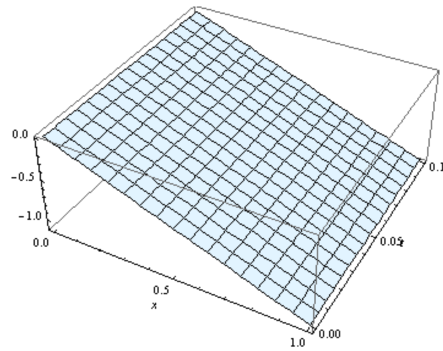

Fig 2, 3D Exact solution of $\zeta(x,t)$ for $t = 0.5$.

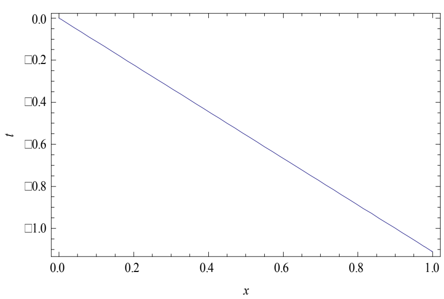

Fig 3, 2D Approximate solution of $\zeta(x,t)$ for $t = 0.5$.

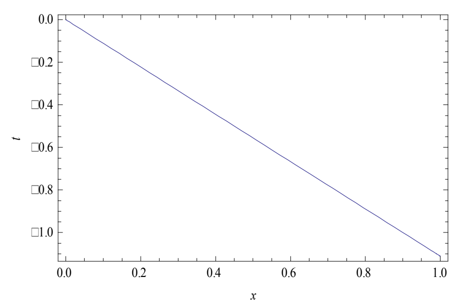

Fig 4, 2D Exact solution for $t = 0.5$.

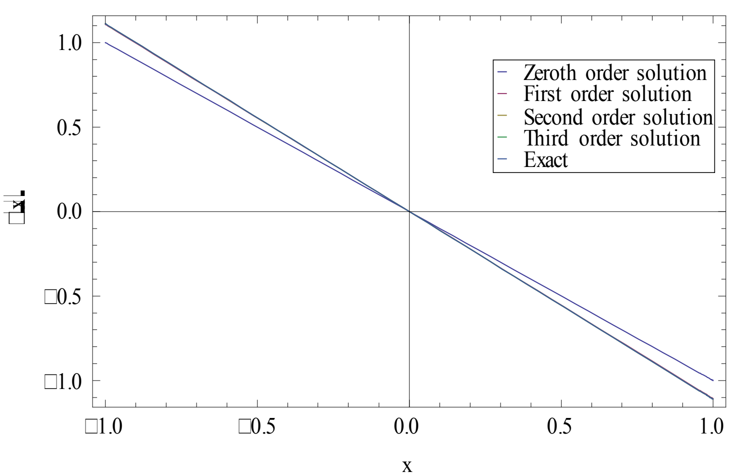

Fig 5, 2D Exact, Zeroth, First, second and Third Order solution of $u(x,t)$ for $t = 0.5$

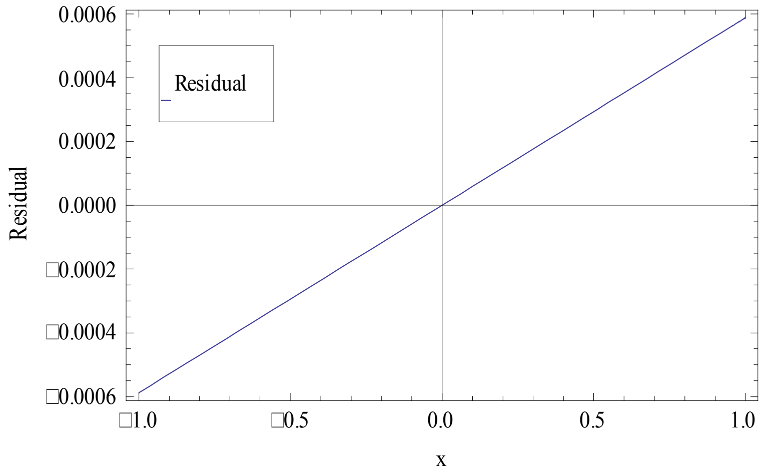

Fig. 6, 2D Residual plot for $t = 0.1$

3. RESULTS, DISCUSSIONS AND CONCLUSION

In Table 1, and Figs. 1-4, we have compared the OHAM results with exact and HPTM results. It is concluded that OHAM results are identical to exact and HPTM results. The absolute errors for spatial domain [0,1] and different t are given in Table 2. The order convergence is given in Table 3 and Fig. 5. The residual of the problem is plotted in Fig. 6. From comparison we have concluded that OHAM is simple easy, flexible, containing less computational, no need of linearization and initial guess when applying to homogeneous nonlinear advection equations.

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