J. Appl. Environ. Biol. Sci., 6(9)38-45, 2016
© 2016, TextRoad Publication

ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

Stability Analysis of a General SEIRS Epidemic Model

Roman Ullah[1], Farhad Ali[1], Muhammad Adil[2], Arshad Ali Shah[2], Zahid Hussain[3], Gul Zaman[4], Muhammad Altaf Khan[4], Saeed Islam[4]

[1] Department of Mathematics, Bacha Khan University Charsadda,Khyber Pakhtunkhwa,Pakistan
[2] Department of Management Sciences, Bacha Khan University Charsadda,Khyber Pakhtunkhwa, Pakistan
[3] Department of Mathematics, University of Malakand, ChakdaraDir, Pakistan
[4] Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, Pakistan

Received: May 5, 2016
Accepted: July 28, 2016

ABSTRACT

In this paper, we developed a general SEIR Sepidemic model that provides knowledge about the occurrence of epidemic. The model can integrate the birth, death and examine the outcome mathematically. Along the way, we show how this simple SEIR Sepidemic model assists to lay a theoretical foundation for public health interventions.
KEYWORDS: mathematical models, SEIR Sepidemic, dynamics

INTRODUCTION

It is proved that mathematical modeling plays an important role in the disease spread and control. A better qualitative assessment can be obtained by an appropriate mathematical model for the problem. Normally mathematical models for epidemics are comprised of system of differential equations that show the rate of change of each interacting component. Numerous advantages of avoiding invasion of infection to population can be obtained by developing a good epidemic model; therefore epidemiological models attracted the attention of many researchers [1-4].Several epidemic models are there in the literature that focuses on the dynamical properties [5-16].In this work, we consider a single host population, the mode of transmission is direct contact, stay in latency period before becoming infectious. The infectious host can be recovered if the required immunity is provided during infectious stage.

In research literature a lot of mathematical models have been presented to study the dynamics of infectious diseases [20, 21, 23]. Khan et al. [21] presented an SEIR epidemic model with preventive vaccination. They divided the host population into four subclasses that is S-susceptible, E-exposed, I-infected and R-recovered. Kaddar et al. [20] proposed a generalized SEIRS epidemic model. The proved the global stability for a generalized SEIRS model by using the geometric approach.

In this work, we present a general SEIRS epidemic model. In our model we assume that the infections stay in the exposed classes before becoming infectious. The term (1- δ) is used which represents the number of individuals that gain natural immunity during the incubation period. Further, we two different transmission $\alpha_1$andα_2, which respectively represent the contact rate between susceptible and exposed, and susceptible and infected individuals. We denote the total host population by $N(t)$, subdividing into four subclasses, that's the susceptible S, latent (exposed) E, Infectious I and recovered R. Thus the total host population can be written as $N(t) = S(t) + E(t) + I(t) + R(t)$.

Model Formulation

This section shows the mathematical formulation of the general infectious SEIR epidemic disease model. The population is categorized is four different subclasses, namely, the susceptible individuals$S(t)$, the individuals latent (Exposed) which are not yet infectious by$E(t)$, infected by $I(t)$ and the individuals whose recover from infection or removed by$R(t)$. Thus, we write,$N(t) = S(t) + E(t) + I(t) + R(t)$ the total size of host population at any time$t$. The model that describes the assumptions above can be written through the following systems of differentials equations:

$$\frac{dS(t)}{dt} = \Lambda - \alpha_1 S(t)E(t) - \alpha_2 S(t)I(t) - \mu S(t) + \gamma R(t), \qquad S(0) = S_o \geq 0$$

***Corresponding Author:** Roman Ullah, Department of Mathematics, Bacha Khan University Charsadda, Khyber Pakhtunkhwa, Pakistan.

$$\frac{dE(t)}{dt} = \alpha_1 S(t)E(t) + \alpha_2 S(t)I(t) - (1-\delta)E(t) - \mu E(t), \qquad \mathrm{E}(0) = E_o \geq 0$$
$$\frac{dI(t)}{dt} = (1-\delta)E(t) - (\omega + \mathcal{E} + \mu)I(t), \mathrm{I}\,(0) = I_o \geq 0 \qquad (1)$$
$$\frac{dR(t)}{dt} = \omega I(t) - \gamma R(t) - \mu R(t), \qquad \mathrm{R}\,(0) = R_o \geq 0.$$

The host population is increased by the recruitment rate Λ, $\alpha_1$and $\alpha_2$respectively show the contact rate between susceptible-exposed and susceptible-infected individuals. The induced death rate is given by ε, natural death rate μ, recovery rate is ω (the recovery may be assumed here, natural or due to treatment). The individuals in the latent class gain immunity naturally at a rate δ while loss at a rate γ. The model (1) has the DFE, denoted by, $E_0 = (S^o, 0, 0, 0)$ and is given by$E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$.

The total dynamics is obtained by summing the equations in (1),

$$\frac{dN}{dt} = \Lambda - \mu N - \varepsilon I \leq \Lambda - \mu N.$$

The feasible region for the model is the closed setΓ, which is positive invariant and bounded, given by
$\Gamma = \left\{(S, E, I, R): 0 \leq S, E, I, R, S + E + I + R \leq \frac{\Lambda}{\mu}\right\}$.

Basic Reproduction Number R_0

This section describes the computation of the basic reproduction number, which is defined as the number of secondary infections generated by single infections when an infection is introduced into a purely susceptible population. The finding of the reproduction number involves the matrices, F and V, see [17]. It follows from [17] that the matrix F and V can be obtained as:

$$\mathcal{F} = \begin{bmatrix} 0 \\ \alpha_1 SE + \alpha_2 SI \\ 0 \\ 0 \end{bmatrix}, \qquad \mathrm{V} = \begin{bmatrix} \alpha_1 SE + \alpha_2 SI - \Lambda + \mu S - \gamma R \\ (1-\delta)E + \mu E \\ -(1-\delta)E + (\omega + \mathcal{E} + \mu)I \\ -\omega I + \gamma R + \mu R \end{bmatrix}.$$

It follows from the disease free equilibrium E_0

$$F = \begin{bmatrix} \frac{\alpha_1 \Lambda}{\mu} & \frac{\alpha_2 \Lambda}{\mu} \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} (1-\delta) + \mu & 0 \\ -(1-\delta) & (\omega + \mathcal{E} + \mu) \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{(1-\delta) + \mu} & 0 \\ \frac{(1-\delta)}{((1-\delta) + \mu)(\omega + \mathcal{E} + \mu)} & \frac{1}{(\omega + \mathcal{E} + \mu)} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\alpha_1 \Lambda}{\mu\left((1-\delta) + \mu\right)} + \frac{\alpha_2 \Lambda (1-\delta)}{\mu\left((1-\delta) + \mu\right)(\omega + \mathcal{E} + \mu)} & \frac{\alpha_2 \Lambda}{\mu(\omega + \mathcal{E} + \mu)} \\ 0 & 0 \end{bmatrix}$$

Thus, the required basic reproduction number for model (1) is given by

$$R_0 = \frac{\alpha_1 \Lambda}{\mu\left((1-\delta) + \mu\right)} + \frac{\alpha_2 \Lambda (1-\delta)}{\mu\left((1-\delta) + \mu\right)(\omega + \mathcal{E} + \mu)}$$
$$= \frac{\alpha_1 \Lambda (\omega + \mathcal{E} + \mu) + \alpha_2 \Lambda (1-\delta)}{\mu\left((1-\delta) + \mu\right)(\omega + \mathcal{E} + \mu)}.$$

The next section describes the local stability of the system (1) at the DFE,E_0.

Local stability:

The present section describes the local stability of the model (1) at the disease free and endemic equilibrium.

Theorem 1: The model (1) is stable locally asymptotically, at the DFEE_0 whenever $R_0 < 1$.

Proof: The proof involves the linearization of the model (1) at DFE E_0 by setting equal to zero the left hand side of (1), which is given by the following Jacobean matrix:

$$J\,(E_0)=\begin{bmatrix} -\mu & -\frac{\alpha_1\Lambda}{\mu} & -\frac{\alpha_2\Lambda}{\mu} & \gamma \\ 0 & \frac{\alpha_1\Lambda}{\mu}-\big((1-\delta)+\mu\big) & \frac{\alpha_2\Lambda}{\mu} & 0 \\ 0 & 1-\delta & -(\omega+\varepsilon+\mu) & 0 \\ 0 & 0 & \omega & -(\gamma+\mu) \end{bmatrix}.$$

We need to show that all the eigenvalues of $J\,(E_0)$ are negative. The first column of $J\,(E_0)$ contains only diagonal element which forms one negative eigenvalue $-\mu$, the other three eigenvalues can be obtained from the matrix $J_1(E_0)$ which is

$$J_1(E_0)=\begin{bmatrix} \frac{\alpha_1\Lambda}{\mu}-\big((1-\delta)+\mu\big) & \frac{\alpha_2\Lambda}{\mu} & 0 \\ (1-\delta\,) & -\,(\omega+\varepsilon+\mu) & 0 \\ 0 & \omega & -\,(\gamma+\mu) \end{bmatrix}$$

Now, again the third column of $J_1(E_0)$ contains only diagonal element which forms negative eigenvalue $-(\gamma+\mu)$, the remaining two eigenvalues can be obtained from the matrix $J_2(E_0)$ which is

$$J_2(E_0)=\begin{bmatrix} \frac{\alpha_1\Lambda}{\mu}-\big((1-\delta)+\mu\big) & \frac{\alpha_2\Lambda}{\mu} \\ (1-\delta) & -\,(\omega+\varepsilon+\mu) \end{bmatrix}$$

The eigenvalues of $J_2(E_0)$ are the roots of the characteristic equation

$$\left(\frac{\alpha_1\Lambda}{\mu}-\big((1-\delta)+\mu\big)-\lambda\right)(-(\omega+\varepsilon+\mu)-\lambda)-\frac{\alpha_2(1-\delta)\Lambda}{\mu}=0$$

$$\Rightarrow \lambda^2+\left((\omega+\varepsilon+\mu)-\left(\frac{\alpha_1\Lambda}{\mu}-\big((1-\delta)+\mu\big)\right)\right)\lambda + \big((1-\delta)+\mu\big)(\omega+\mathcal{E}+\mu)\left(1-\frac{\alpha_2(1-\delta)\Lambda+\alpha_1 v(\omega+\varepsilon+\mu)}{\mu(\omega+\varepsilon+\mu)\big((1-\delta)+\mu\big)}\right)=0$$

$$\Rightarrow \lambda^2+\left((\omega+\varepsilon+\mu)-\left(\frac{\alpha_1\Lambda}{\mu}-\big((1-\delta)+\mu\big)\right)\right)\lambda+\big((1-\delta)+\mu\big)(\omega+\mathcal{E}+\mu)(1-R_0)=0$$

$$\Rightarrow A_2\lambda^2+A_1\lambda+A_0=0,$$

Where $A_2=1,\qquad A_1=(\omega+\varepsilon+\mu)-\left(\frac{\alpha_1\Lambda}{\mu}-\big((1-\delta)+\mu\big)\right),$

$$A_0=\big((1-\delta)+\mu\big)(\omega+\mathcal{E}+\mu)(1-R_0).$$

The above quadratic equations will give two negative eigenvalues if and only if $R_0<1$ and $+\varepsilon+\mu+(1-\delta)+\mu>\frac{\alpha_1\Lambda}{\mu}$. We see that in the above polynomial $A_2=1, A_1$ will be positive only when $\omega+\varepsilon+\mu+(1-\delta)+\mu>\frac{\alpha_1\Lambda}{\mu}$ and A_0 will be positive if $R_0<1$. Thus for these two conditions all the roots of the polynomial will be negative. Hence the model (1) at the DFE E_0 is stable locally asymptotically whenever $R_0<1$ and $\omega+\varepsilon+\mu+(1-\delta)+\mu>\frac{\alpha_1\Lambda}{\mu}$.

Endemic Equilibrium

The endemic equilibria of the model (1) at endemic equilibrium $E^1=(S^*,E^*,I^*,R^*)$ is given by

$$S^*=\frac{(1-\delta+\mu)(\grave{o}+\mu+\omega)}{(\grave{o}+\mu+\omega)\alpha_1+(1-\delta)\alpha_2}\;,E^*=\frac{I^*(\grave{o}+\mu+\omega)}{(1-\delta)}\;,R^*=\frac{\omega I^*}{\gamma+\mu},$$

$$I^*\;=\frac{\mu(1-\delta+\mu)(\grave{o}+\mu+\omega)(1-\delta)(\gamma+\mu)(R_0-1)}{\big((\grave{o}+\mu+\omega)\alpha_1+(1-\delta)\alpha_2\big)\big((\gamma+\mu)(1-\delta+\mu)(\grave{o}+\mu)+\mu(1+\gamma-\delta+\mu)\omega\big)}.$$

A unique positive endemic equilibrium exists if and only if $R_0>1$.

The following theorem analyzes the local stability of the endemic equilibrium.

Theorem 2: The model (1) at the endemic equilibrium E^1 is stable locally asymptotically if $R_0 > 1$ and the conditions of Routh-Hurwitz criteria is satisfied.
Proof: At the endemic equilibrium E^1we obtain the following jacobian matrix,

$$J\left(E^1\right) = \begin{bmatrix} -(\alpha_1 E^* + \alpha_2 I^* + \mu) & -\alpha_1 S^* & -\alpha_2 S^* & \gamma \\ \alpha_1 E^* + \alpha_2 I^* & \alpha_1 S^* - (1-\delta) - \mu & \alpha_2 S^* & 0 \\ 0 & 1-\delta & -(\omega + \varepsilon + \mu) & 0 \\ 0 & 0 & \omega & -(\gamma + \mu) \end{bmatrix}$$

The Jacobian matrix $J\left(E^1\right)$has the following characteristics equation:

$$\lambda^4 + l_1\lambda^3 + l_2\lambda^2 + l_3\lambda + + l_3 = 0,$$

Where

$$l_2 = \dot{o} + 3\mu + \omega + 3\mu\left(\dot{o} + 2\mu + \omega\right) - \delta\left(\dot{o} + 3\mu + \omega\right) + \gamma\left(1 - \delta + \dot{o} + 3\mu + \omega\right)$$
$$+\left(-S^*\left(\gamma + \dot{o} + 3\mu + \omega\right) + E^*\left(1 + \gamma - \delta + \dot{o} + 3\mu + \omega\right)\right)\alpha_1$$
$$+I^*\alpha_2 + \left(s\left(-1 + \delta\right) + I^*\left(\gamma - \delta + \dot{o} + 3\mu + \omega\right)\right)\alpha_2$$

$$l_3 = \mu\left(\dot{o}\left(2 - 2\delta + 3\mu\right) + \mu\left(3 - 3\delta + 4\mu\right) + \left(2 - 2\delta + 3\mu\right)\omega\right) + \gamma\left(3\mu^2 + \dot{o}\left(1 - \delta + 2\mu\right) + \omega(1-\delta) + 2\mu\left(1 - \delta + \omega\right)\right)$$
$$+(-S^*\left(\gamma\left(\dot{o} + 2\mu + \omega\right) + \mu\left(2\dot{o} + 3\mu + 2\omega\right)\right)$$
$$+E^*\left(\dot{o} + 2\mu + \omega - \delta\left(\dot{o} + 2\mu + \omega\right) + \gamma\left(1 - \delta + \dot{o} + 2\mu + \omega\right) + \mu\left(2\dot{o} + 3\mu + 2\omega\right)\right))\alpha_1$$
$$+\left(S^*\left(-1 + \delta\right)\left(\gamma + 2\mu\right) + I^*\left(\dot{o} + 2\mu + \omega - \delta\left(\dot{o} + 2\mu + \omega\right) + \gamma\left(1 - \delta + \dot{o} + 2\mu + \omega\right) + \mu\left(2\dot{o} + 3\mu + 2\omega\right)\right)\right)\alpha_2$$

$$l_4 = \left(E^*\left(\gamma + \mu\right)\left(1 - \delta + \mu\right)\left(\dot{o} + \mu\right) + E^*\mu\left(1 + \gamma - \delta + \mu\right)\omega - S^*\mu\left(\gamma + \mu\right)\left(\dot{o} + \mu + \omega\right)\right)\alpha_1 +$$
$$\left(S^*\left(-1 + \delta\right)\mu\left(\gamma + \mu\right) + I^*\left(\gamma + \mu\right)\left(1 - \delta + \mu\right)\left(\dot{o} + \mu\right) + I^*\mu\left(1 + \gamma - \delta + \mu\right)\omega\right)\alpha_2 + \left(1 - \delta + \mu\right)\mu\left(\gamma + \mu\right)\left(\dot{o} + \mu + \omega\right)$$

The characteristics equation above will give four eigenvalues with negative real parts if and only the conditions of Routh-Hurtwiz criteria: that is, the coefficients l_1, l_2, l_3 and l_4 are positive and$l_1 l_2 l_3 - l_3^2 - l_1^2 l_4 > 0$. Thus, Routh-Hurtwiz criteria ensures the system (1) at endemic equilibrium $E^1$is stable locally asymptotically whenever $R_0 > 1$ and the Routh-Hurtwiz criteria is satisfied.

Global Dynamics

In the given section we reduce the model (1) by using $S + E + I + R = N = 1$, and making the assumptions$R = (1 - S - E - I)$, and then we obtain the following reduced model:

$$\frac{dS(t)}{dt} = \Lambda - \alpha_1 S(t)E(t) - \alpha_2 S(t)I(t) - \mu S(t) + \gamma(1 - S(t) - E(t) - I(t)), \qquad S(0) = S_o \geq 0$$
$$\frac{dE(t)}{dt} = \alpha_1 S(t)E(t) + \alpha_2 S(t)I(t) - (1-\delta)E(t) - \mu E(t), \qquad E(0) = E_o \geq 0$$
$$\frac{dI(t)}{dt} = (1-\delta)E(t) - (\omega + \mathcal{E} + \mu)I(t), I(0) = I_o \geq 0 \qquad (2)$$

The DFE and EE of the model (2) is now denoted by $E_2 = (\frac{\Lambda+\gamma}{\mu+\gamma},\ 0,\ 0)$ and $E_3 = (S^*,\ E^*,\ I^*)$. For the global dynamics we will study the model (2).We follow [24] to present the global stability of system (2). We rewrite the model (2) in the following form

$$\frac{dY}{dt} = F(Y, V)$$
$$\frac{dV}{dt} = G(Y, V), G(Y, 0) = 0,$$

where $Y = S$ and $Z = (V,\ I)$ respectively denotes the population of uninfected (susceptible) and infected individuals (exposed and infected) with $X \in \mathbb{R}$ and $Z \in \mathbb{R}^2$. The model (2) will be stable globally asymptotically when the conditions given in the following are hold.

C_1 For $\frac{dY}{dt} = F(Y,\ 0) = 0, Y_0$ is stable globally asymptotically.

C_2 $G(Y,V) = LV - \hat{G}(Y,V)$, where $\hat{G}(Y,V) \geq 0$, for $(Y,V) \in \Gamma$,

where $L = D_L G(Y_0, 0)$, shows an M-matrix and $\Gamma$is the biologically feasible region. Following the method in [24], we present the following theorem for the global stability of DFE of the model (2).

Theorem 3: The DFE of the model (2) is stable globally asymptotically whenever $R_0<1$.

Proof: Choose $Y = S$ and $V = (E,\ I)$ and U= $(Y_0, 0)$, where $Y_0 = S_0 = \frac{\Lambda+\gamma}{\mu+\gamma}$.

The conditions mentioned above can be applied to model (2) as:

$\frac{dY}{dt} = F(Y,0) = \Lambda - \mu S_0 + \gamma(1 - S_0)$ which is stable globally asymptotically when $t \longrightarrow \infty$.

$$G(Y,V) = LV - \hat{G}(Y,V) = \begin{bmatrix} (1-\delta+\mu)+\alpha_1 S_0 & \alpha_2 S_0 \\ (1-\delta) & -(\omega+\mathcal{E}+\mu) \end{bmatrix}\begin{bmatrix} E \\ I \end{bmatrix} - \begin{bmatrix} \alpha_1 E(S_0-S)+\alpha_2 I(S_0-S) \\ 0 \end{bmatrix},$$

where $= \begin{bmatrix} (1-\delta+\mu)+\alpha_1 S_0 & \alpha_2 S_0 \\ (1-\delta) & -(\omega+\mathcal{E}+\mu) \end{bmatrix}$, $V = \begin{bmatrix} E \\ I \end{bmatrix}$ and $\hat{G}(Y,V) = \begin{bmatrix} \alpha_1 E(S_0-S)+\alpha_2 I(S_0-S) \\ 0 \end{bmatrix}$.

In model (2) the total population is bounded by $S_0 = \frac{\Lambda+\gamma}{\mu+\gamma}$, that is $S,\ E,\ I \leq S_0$, where S_0 represents the DFE of the model (2) and hence $\hat{G}(Y,V) \geq 0$. Thus the two conditions presented above are satisfied. Thus, we can conclude that the DFE of the model (2) is stable globally asymptotically.

Global stability of Endemic Equilibrium

This section describes the global stability of the endemic equilibrium of the model (2). For the proof we use the geometric approach method [19]. Many authors used this method in his papers, see [20-23].

Theorem: The endemic equilibrium of the reduced model (2) is globally asymptotically stable if $R_0 > 1$.

Proof: The endemic equilibrium of the model (2) is given by

$$J^* = \begin{bmatrix} -\mu - E\alpha_1 - I\alpha_2 & -\alpha_1 S & -\alpha_2 S \\ E\alpha_1 + I\alpha_2 & -1+\delta-\mu+\alpha_1 S & \alpha_2 S \\ 0 & 1-\delta & -\epsilon-\mu-\omega \end{bmatrix},$$

The second additive compound matrix associated to $J^*$is

$$J^{[2]} = \begin{bmatrix} f_1 & S\alpha_2 & S\alpha_2 \\ (1-\delta) & f_2 & -S\alpha_1 \\ 0 & E\alpha_1+I\alpha_2 & f_3 \end{bmatrix}$$

$f_1 = -\mu - E\alpha_1 - I\alpha_2 + -1 + \delta - \mu + S\alpha_1, f_2 = -\mu - E\alpha_1 - I\alpha_2 - (\epsilon+\mu+\omega), f_3 = -1 + \delta - \mu + S\alpha_1 - (\epsilon+\mu+\omega)$.

Choose the function $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{E}{I} & 0 \\ 0 & 0 & \frac{E}{I} \end{bmatrix}$, and $H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{I}{E} & 0 \\ 0 & 0 & \frac{I}{E} \end{bmatrix}$, $H_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-IE'+EI'}{I^2} & 0 \\ 0 & 0 & \frac{-IE'+EI'}{I^2} \end{bmatrix}$,

So that $H_f H^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{E'}{E}-\frac{I'}{I} & 0 \\ 0 & 0 & \frac{E'}{E}-\frac{I'}{I} \end{bmatrix}$. Then $HJ^{[2]}H^{-1} =$

$$\begin{bmatrix} -1+\delta-2\mu-E\alpha_1+S\alpha_1-I\alpha_2 & \frac{\alpha_2 S I}{E} & \frac{\alpha_2 S I}{E} \\ \frac{(1-\delta)E}{I} & -\epsilon-2\mu-\omega-E\alpha_1-I\alpha_2 & -S\alpha_1 \\ 0 & E\alpha_1+I\alpha_2 & -1+\delta-\epsilon-2\mu-\omega+S\alpha_1 \end{bmatrix}.$$

So $M = H_f H^{-1} + HJ^{[2]}H^{-1} = \begin{bmatrix} \boldsymbol{f_{11}} & \frac{\alpha_2 S I}{E} & \frac{\alpha_2 S I}{E} \\ \frac{(1-\delta)E}{I} & \boldsymbol{f_{22}} & -S\alpha_1 \\ 0 & E\alpha_1+I\alpha_2 & \boldsymbol{f_{33}} \end{bmatrix}$

where

$\boldsymbol{f_{11}} = -1 + \delta - 2\mu - E\alpha_1 + S\alpha_1 - I\alpha_2, \boldsymbol{f_{22}} = -\epsilon - 2\mu - \omega - E\alpha_1 - I\alpha_2 + \frac{E'}{E} - \frac{I'}{I}$,

$$\boldsymbol{f_{33}} = -1 + \delta - \epsilon - 2\mu - \omega + S\alpha_1 + \frac{E'}{E} - \frac{I'}{I}.$$

Let $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, where $M_{11} = -1 + \delta - 2\mu - E\alpha_1 + S\alpha_1 - I\alpha_2, M_{12} = \max[\frac{\alpha_2 S I}{E}, \frac{\alpha_2 S I}{E}]$

$M_{21} = \left[\frac{(1-\delta)E}{I}, 0\right]^T, M_{22} = \begin{bmatrix} -\epsilon - 2\mu - \omega - E\alpha_1 - I\alpha_2 + \frac{E'}{E} - \frac{I'}{I} & -S\alpha_1 \\ E\alpha_1 + I\alpha_2 & -1 + \delta - \epsilon - 2\mu - \omega + S\alpha_1 + \frac{E'}{E} - \frac{I'}{I} \end{bmatrix}$.

Now consider the norm in R^3 as $|(m_1, m_2, m_3)| = \max\{|m_1|, |m_2| + |m_3|\}$, where (m_1, m_2, m_3) represent the vector in R^3. The Lozinski associated to the above norm is shown by χ. Thus it follows from [18]:
$\chi(M) \leq sup\{b_1, b_2\} = sup\{\chi_1(M_{11}) + |M_{12}|, \chi_1(M_{22}) + |M_{21}|\}$.
Therefore

$$b_1 = \chi_1(M_{11}) + |M_{12}| = -(1 - \delta + \mu) - \mu - E\alpha_1 + S\alpha_1 - I\alpha_2 + \frac{\alpha_2 S I}{E} \leq \frac{E'}{E} - \mu - I\alpha_2 - E\alpha_1 \leq \frac{E'}{E} - \mu.$$

Using the fact $\frac{E'}{E} = \frac{\alpha_2 S I}{E} + S\alpha_1 - (1 - \delta + \mu)$.

And $b_2 = \chi_1(M_{22}) + |M_{21}| = \max\left\{-(\epsilon + \mu + \omega) - \mu + \frac{E'}{E} - \frac{I'}{I}, -(1 - \delta + \mu) - \mu - (\epsilon + \mu + \omega + \frac{E'}{E} - \frac{I'}{I}\right\} + \frac{(1-\delta)E}{I} \leq \frac{E'}{E} - \frac{I'}{I} - \mu + \frac{(1-\delta)E}{I} - (\epsilon + \mu + \omega) \leq \frac{E'}{E} - \mu$.

Using the fact $\frac{I'}{I} = \frac{(1-\delta)E}{I} - (\epsilon + \mu + \omega)$.

So, $\chi(M) = sup\{b_1, b_2\} = \frac{E'}{E} - \mu$.

Every solution (S(t), E(t), I(t)) of proposed system (2) with S(0), E(0), I(0) belong to some compact absorbing set (say Θ). It follows

$$\chi(M) = sup\{b_1, b_2\} = \frac{E'}{E} - \mu = \frac{1}{t}\int_0^t \chi(M)ds \leq \frac{1}{t} ln \frac{E(t)}{E(0)} - \mu \leq -\frac{\mu}{2}.$$

Numerical results

We find the numerical solution of the proposed model (1) by choosing the base line for the susceptible population S=50, Exposed population E=10, Infected population I=10, Recovered population R=10. The parameters and their values are given as$\mu = 0.01$, $\Lambda = 0.05$, $\alpha_1 = 0.005$, $\alpha_2 = 0.0025$, $\varepsilon = 0.078$, $\omega = 0.2$, $\gamma = 0.4$ *and* $\delta = 0.4$. Figure 1 shows the behavior of distinct classes of the model.

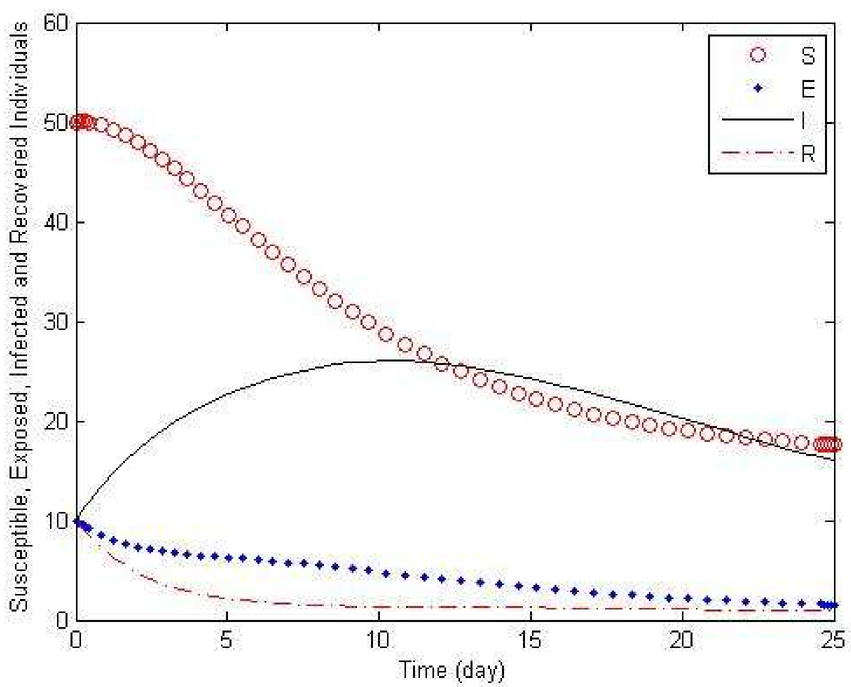

Figure 1: Dynamical behavior of the proposed model.

Conclusion

In this work, we studied a general SEIRS epidemic model of infectious disease. The transmission rate between susceptible-exposed and susceptible-infected was assumed. We investigated that the model is stable at the infection free state when the associated basic reproduction number less than unity. A stable endemic equilibrium was obtained for the case when the basic reproduction number exceeds than unity. Further, the stability of the reduced model was investigated. The disease free stability is examined by Castillo-Chavez method. Using the geometric approach method, the endemic equilibrium of the reduced model is derived, which is found to be stable globally asymptotically when the basic reproduction number exceeds than unity.

REFERENCES

[1] MA Khan, Y Khan, S Khan, S Islam, Global stability and vaccination of an SEIVR epidemic model with saturated incidence rate, International Journal of Biomathematics, 1650068, 2016.

[2] Ullah, R; Zaman, G; Islam, S; “stability analysis of a general SIR epidemic model” VFAST Transaction on Mathematics2013(1) 16-20.

[3] Guihua, L; Zhen, J; “Global stability of an SEI epidemic model with general contact rate,” Chaos, Solitons and Fractals,2005 (5) 997–1004.

[4] V. Wiwanitkit; "Unusual mode of transmission of dengue," The Journal of Infection inDeveloping Countries, 2009 (30) 51--54,

[5] C. J. Sun, Y. P. Lin and S. P. Tang, “Global stability for an special SEIR epidemic model with nonlinear incidence rates,” Chaos, Solitons and Fractals,2007(33) 290-297.

[6] Y. Nakata and T. Kuniya, “Global dynamics of a class of SEIRS epidemic models in a periodic environment,” Journal of MathematicalAnalysis and Applications,2010 (10) 230-237.

[7] G. H. Li and Z. Jin, “Global stability of an SIR epidemic model with nonlinear incidence rates,” Chaos, Solitons &Fractals,2007(14) 142-159.

[8] M. Y. Li and J. S. Muldowney, “Global stability for the SEIR model in epidemiology,” Mathematical Biosciences,1995 (125) 155-164

[9] B. K. Mishra and N. Jha, “SEIQRS model for the transmission of malicious objects in computer network,” Applied Mathematical Modelling, 2010 (34) 710-715.

[10] J. Hou and Z. D. Teng, “Continuous and implusive vaccination of SEIR epidemic models with saturation incidence rates,” Mathematics and Computers in Simulation, 2009 (79) 3038-3054.

[11] X. Z. Meng, J. J. Jiao and L. S. Chen, “Two profitless delays for an SEIRS epidemic disease model with vertical transmission and pulse vaccination,” Chaos, Solitons &Fractals, 2009 (40) 2114-2125.

[12] X. Wang, Y. D. Tao and X. Y. Song, “Pusle vaccination on SEIR epidemic model with nonlinear incidence rate,” Applied Mathematics and Computation, 2009 (210) 398-404.

[13] X. Z. Li and L. L. Zhou, “Global stability of an SEIR epidemic model with vertical transmission and saturating contact rate,” Chaos, Solitons&Fractals, 2009 (40) 874-884.

[14] R. Ullah, G. Zaman, S. Islam and I. Ahmad, “Dynamical features and vaccination strategies in an SEIR epidemic model”, *Research Journal of Recent Sciences*, 2 (10) (2013) 48-56.

[15] R. Ullah, G. Zaman, S. Islam, “Prevention of influenza Pandemic by Multiple control strategiesV,*J.Appl. Math.*, vol 2012, Article ID 294275, 14 pages, 2012. doi:10.1155/2012/294275.

[16] R. Ullah, G. Zaman, S. Islam, “Multiple Control Strategies for Prevention of Avian Influenza PandemicV *The Scientific World Journal*, Volume 2014, Article ID 949718, 9 pages, http//dx.doi.org/10.1155/2014/949718.

[17] P. van den Driessche and J. Watmough, “Reproductive numbers and sub-threshold endemic equilibria for compartment models of disease trsmission, Math. Biosci., 18 (2001) 29–48.

[18] R.H. Martin, Logarithmic norms and projections applied to linear differential systems, J. Math. Anal. Appl. (1974) 432–454.

[19] M.Y. Li, J.S. Muldowney, A geometric approach to global-stability problems, SIAM J. Math. Anal. 27 (4) (1996) 1070–1083.

[20] Abdelilah Kaddar, Soufiane Elkhaiar, Fatiha Eladnani, Global Asymptotic Stability of a Generalized SEIRS Epidemic Model, Differ Equ Dyn Syst (2016). doi:10.1007/s12591-016-0313-y

[21]MA Khan, Q Badshah, S Islam, I Khan, S Shafie, SA Khan, Global dynamics of SEIRS epidemic model with non-linear generalized incidences and preventive vaccination, Advances in Difference Equations 2015 (1), 1-18.

[22]MA Khan, S Islam, SA Khan, G Zaman, Global stability of vector-host disease with variable population size, BioMed research international 2013, 1-10.

[23]MA Khan, S Ullah, DL Ching, I Khan, S Ullah, S Islam, T Gul, A Mathematical Study of an Epidemic Disease Model Spread by Rumors, Journal of Computational and Theoretical Nanoscience 13 (5), 2856-2866.

[24] C. Castillo-Chavez, Z. Feng, W. Huang, Mathematical Approaches for Emerging and Re-Emerging Infectious Diseases: An Introduction, Springer Verlag, 2002.