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Numerical Solution of Fractional Telegraph Equation Using the Second Kind Chebyshev Wavelets Method

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ABSTRACT

In this paper, the two-dimensional second kind Chebyshev wavelets are applied for numerical solution of the time-fractional telegraph equation with Dirichlet boundary conditions. In this way, a new operational matrix of fractional derivative for the second wavelets is derived and then this operational matrix has been employed to obtain the numerical solution of the above mentioned problem. The main characteristic behind this approach is that it reduces such problems to those of solving a system of algebraic equations which greatly simplifying the problem. The power of this manageable method is illustrated.

KEYWORDS: Second kind Chebyshev wavelet, Telegraph equation, Dirichlet boundary condition, Fractional derivative.

1. INTRODUCTION

In recent years, fractional calculus and differential equations have found enormous applications in mathematics, physics, chemistry and engineering because of this fact that, a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. The application of the fractional calculus have been demonstrated by many authors. For examples, it is applied to model the nonlinear oscillation of earthquakes [1], fluid-dynamic traffic [2], frequency dependent damping behavior of many viscoelastic materials [3], continuum and statistical mechanics [4], colored noise [5], solid mechanics [6], economics [7], signal processing [8], and control theory [9]. However, during the last decade fractional calculus has attracted much more attention of physicist and mathematicians. Due to the increasing applications, some schemes have been proposed to solve fractional differential equations. The most frequently used methods are Adomian decomposition method (ADM) [10-12], homotopy perturbation method [13], homotopy analysis method [14], Variational iteration method (VIM) [15, 16], Fractional differential transform Method (FDTM) [17-22], Fractional difference method (FDM) [23], power series method [24], generalized block pulse operational matrix method [25] and Laplace transform method [26]. Also, recently the operational matrices of fractional order integration for the Haar wavelet [27], Legendre wavelet [28] and the Chebyshev wavelets of first kind [29, 30] and second kind [31] have been developed to solve the fractional differential equations.

In this paper consider the time-fractional telegraph equation of order $\alpha\,(1<\alpha\le 2)$ as:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)+\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}u(x,t)+u(x,t)=\frac{\partial^{2}}{\partial x^{2}}u(x,t)+f(x,t),\quad (x,t)\in[0,1]\times[0,1]. \tag{1}$$

where $\frac{\partial^{\beta}}{\partial t^{\beta}}$ denotes the Caputo fractional derivative of order β, that will be described in the next section. This equation is commonly used in the study of wave propagation of electric signals in a cable transmission line and also in wave phenomena. This equation has been also used in modeling the reaction-diffusion processes in various branches of engineering sciences and biological sciences by many researchers (see [32] and references therein).

The main purpose of this paper is to apply the second kind Chebyshev wavelets for solving time-fractional telegraph equation (1). In this way, we first describe some properties of the second kind Chebyshev polynomials and Chebyshev wavelets. Then, a new operational matrix of fractional derivative for the second kind Chebyshev wavelets are derived and are applied to obtain approximate solution for the under study problem. This paper is organized as follows: In Section 2, some necessary definitions of the fractional calculus are reviewed. In Section 3, the second kind Chebyshev polynomials and the second kind Chebyshev wavelets with some useful theorems are investigated. In Section 4, the proposed method is described. In Section 5, some numerical examples are presented. Finally a conclusion is drawn in Section 6.

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2. Basic definitions

In the development of theories of fractional derivatives and integrals, many definitions for fractional derivatives and integrals are appeared, such as Riemann-Liouville and Caputo [33], which are described below:

Definition 2-1

A real function $u(x), x>0$, *is said to be in the space* $C_\mu, \mu \in \mathsf{R}$ *if there exists a real number* $p\,(>\mu)$ *such that* $u(x)=x^p u_1(x)$, *where* $u_1(x)\in C[0,\infty]$ *and it is said to be in the space* C_μ^n *if* $u^{(n)}\in C_\mu, n\in \mathsf{N}$.

Definition 2-2

The Riemann-Liouville fractional integration operator of order $\alpha \ge 0$ *of a function* $u\in C_\mu, \mu\ge -1$, *is defined as [33]:*

$$\left(I^\alpha u\right)(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x (x-t)^{\alpha-1}u(t)dt, & \alpha>0,\\ u(x), & \alpha=0.\end{cases} \tag{2}$$

It has the following properties:

$$\left(I^\alpha I^\beta u\right)(x)=\left(I^{\alpha+\beta}u\right)(x),\quad I^\alpha x^\vartheta=\frac{\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)}x^{\alpha+\vartheta}, \tag{3}$$

where $\alpha,\beta\ge 0$ and $\vartheta>-1$.

Definition 2-3

The fractional derivative operator of order $\alpha>0$ *in the Caputo sense is defined as [33]:*

$$\left(D_*^\alpha u\right)(x)=\begin{cases}\dfrac{d^n u(x)}{dx^n}, & \alpha=n\in \mathsf{N},\\ \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x (x-t)^{n-\alpha-1}u^{(n)}(t)dt, & n-1<\alpha<n.\end{cases} \tag{4}$$

where n is an integer, $x>0$, and $u\in C_1^n$.

The useful relation between the Riemann-Liouvill operator and Caputo operator is given by the following expression:

$$\left(I^\alpha D_*^\alpha u\right)(x)=u(x)-\sum_{j=0}^{n-1}u^{(j)}(0^+)\frac{x^j}{j!},\ x>0,\quad n-1<\alpha\le n, \tag{5}$$

where n is an integer, $x>0$, and $u\in C_1^n$.

For more details about fractional calculus see [33].

3. The second kind Chebyshev polynomials and wavelets

The well-known second kind Chebyshev polynomial $U_m(z)$ form a complete set of orthogonal functions with respect to the weight function $w(z)=\sqrt{1-z^2}$ on the interval $[-1,1]$. They can be determined with the aid of the following recurrence formula [34]:

$$U_m(z)=2zU_{m-1}(z)-U_{m-2}(z),\quad n=2,3,\ldots,$$

with $U_0(z)=1$ and $U_1(z)=2z$. For practical use of these polynomials on the interval of interest $[0,1]$, it is necessary to shift the defining domain by means of the following substitution:

$$z=2x-1,\quad 0\le x\le 1.$$

So, the shifted second kind Chebyshev polynomials $U_m^*(x)$ are obtained on the interval $[0,1]$ as $U_m^*(x)=U_m(2x-1)$. The orthogonality condition for these shifted polynomials is:

$$\int_0^1 U_m^*(x)U_n^*(x)\sqrt{1-(2x-1)^2}\,dx=\frac{\pi}{4}\delta_{mn}, \tag{6}$$

where δ_{mn} is the Kroneker delta.

The analytic form of the shifted second kind Chebyshev polynomial is:

$$U_m^*(x) = \sum_{i=0}^{m} a_{mi}, \tag{7}$$

where

$$a_{mi} = (-1)^{m-i} \frac{(m+i+1)! 2^{2i}}{(m-i)!(2i+1)!} x^i. \tag{8}$$

The second kind Chebyshev wavelets $\psi_{nm}(x) = \psi(k,n,m,x)$, which is constructed from it's corresponding polynomials involve four arguments, $n = 1,\ldots,2^k$, k is assumed any positive integer, m is the degree of the second kind Chebyshev polynomials and the variable x is defined over $[0,1]$. They are defined on the interval $[0,1]$ as [31]:

$$\psi_{nm}(x) = \begin{cases} \sqrt{\frac{2}{\pi}} 2^{\frac{k+1}{2}} U_m(2^{k+1}x - 2n + 1), & x \in [\frac{n-1}{2^k}, \frac{n}{2^k}], \\ 0, & o.w. \end{cases} \tag{9}$$

We should note that in dealing with the second kind Chebyshev wavelets the weight function $w^*(x) = w(2x-1)$ have to be dilate and translate as $w_n(x) = w(2^{k+1}x - 2n + 1)$.

A function $u(x)$ defined on $[0,1]$ may be expanded by the second kind Chebyshev wavelets as:

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \tag{10}$$

where $c_{nm} = (u(x), \psi_{nm}(x))_{w_n}$, and $(.,.)$ denotes the inner product in $L^2_{w_n}[0,1]$.

If the infinite series in (10) is truncated, then it can be written as:

$$u(x); \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = C^T \Psi(x), \tag{11}$$

where C and $\Psi(x)$ are $\hat{m} = 2^k M$ column vectors given by:

$$C = \left[c_1, c_2, \ldots, c_M \mid c_{M+1}, \ldots, c_{2M} \mid, \ldots, \mid c_{(2^k-1)M+1}, \ldots, c_{\hat{m}}\right]^T,$$

and

$$\Psi(x) = \left[\Psi_1(x), \ldots, \Psi_M(x) \mid \Psi_{M+1}(x), \ldots, \Psi_{2M}(x) \mid, \ldots, \mid \Psi_{(2^k-1)M+1}(x), \ldots, \Psi_{\hat{m}}(x)\right]^T, \tag{12}$$

in which $c_i = c_{nm}$, $\Psi_i(x) = \psi_{nm}(x)$. The index i, is determined by the relation $i = (n-1)M + m + 1$.

Similarly, an arbitrary function of two variables $u(x,t)$ defined over $[0,1] \times [0,1]$, may be expanded into second kind Chebyshev wavelets basis as:

$$u(x,t); \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} u_{ij} \psi_i(x) \psi_j(y) = \Psi^T(x) U \Psi(t), \tag{13}$$

where $U = [u_{ij}]$ and $u_{ij} = \left(\psi_i(x), \left(u(x,t), \psi_j(t)\right)\right)$.

4. The operational matrix of fractional derivative

Here, we present a procedure to derive the operational matrix of fractional derivative in the Caputo sense for the second kind Chebyshev wavelets.

Remark 1 *By using the shifted second kind Chebyshev polynomials, any component* $\psi_{nm}(x)$ *of (12) can be written as:*

$$\psi_{nm}(x) = \sqrt{\frac{2}{\pi}} 2^{\frac{k+1}{2}} U_m^*(2^k x - n) \chi_{I_{nk}}(x),$$

where $I_{nk} = [\frac{n}{2^k}, \frac{n+1}{2^k}]$, $n = 0,\ldots,2^k - 1, m = 0,1,\ldots,M-1$ and $\chi_{I_{nk}}(x)$ is the characteristic function defined as:

$$\chi_{I_{nk}}(x)=\begin{cases}1, & x\in[\frac{n}{2^k},\frac{n+1}{2^k}],\\ 0, & o.w.\end{cases}$$

Next we present a useful theorem about fractional derivative of the second kind Chebyshev wavelets:

Lemma 4-1 *Let* $\psi_{nm}(x)$ *be a component of (12) defined on the interval* I_{nk}*, and* $D_*^{\alpha}\psi_{nm}(x)$ *be fractional derivative of order* $\alpha>0(\lceil\alpha\rceil-1<\alpha<\lceil\alpha\rceil)$*, of* $\psi_{nm}(x)$ *with respect to* x*. Then for any* $n=0,1,\ldots,2^k-1$*, we have:*

$$D_*^{\alpha}\psi_{nm}(x)=\begin{cases}0, & 0\le m<\lceil\alpha\rceil,\\ \sum_{j=0}^{M-1}\Omega_{\alpha}^{(n)}(m,j)\psi_{nj}(x)+\sum_{l=n+1}^{2^k-1}\sum_{j=0}^{M-1}\hat{\Omega}_{\alpha}^{(n)}(m,j)\psi_{lj}(x), & m\ge\lceil\alpha\rceil,\end{cases}$$

where

$$\Omega_{\alpha}^{(n)}(m,j)=\sum_{i=\lceil\alpha\rceil}^{m}\frac{2^{k\alpha+2(i+1)}(-1)^{m+j-i}(j+1)(i)!(m+i+1)!\Gamma(j-i+\alpha)\Gamma(i-\alpha+\frac{3}{2})}{\sqrt{\pi}(m-i)!(2i+1)!\Gamma(\alpha-i)\Gamma(i-\alpha+1)\Gamma(j+i-\alpha+3)},$$

and

$$\hat{\Omega}_{\alpha}^{(n)}(m,j)=\sum_{i=\lceil\alpha\rceil}^{m}\frac{2^{k(i+1)+2}b_{mi}(i)!}{\pi\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(i-\lceil\alpha\rceil+1)}\times\sum_{r=0}^{j}b_{jr}\int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}}f_i(x)(2^kx-l)^r w_l^*(x)dx,$$

b_{mi} and b_{jr} are defined in (8) and

$$f_i(x)=\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}\frac{(s-\frac{n}{2^k})^{i-\lceil\alpha\rceil}}{(x-s)^{\alpha-\lceil\alpha\rceil+1}}ds\,\chi_{[\frac{n+1}{2^k},1]}(x).$$

Proof. For $m<\lceil\alpha\rceil$, from definitions of the shifted second kind Chebyshev polynomials and Caputo's derivative the statement is clear. For $m\ge\lceil\alpha\rceil$, from remark (1) and analytic form of the shifted second kind Chebyshev polynomials we have:

$$\psi_{nm}(x)=\sqrt{\frac{2}{\pi}}2^{\frac{k+1}{2}}\sum_{i=0}^{m}b_{mi}(2^kx-n)^i\chi_{I_{nk}}(x). \tag{14}$$

We notice that this function is zero outside the interval $[\frac{n}{2^k},\frac{n+1}{2^k}]$. Now by applying D_*^{α} on both sides of (14) we get:

$$D_*^{\alpha}\psi_{nm}(x)=\sum_{i=0}^{m}a_{mi}D_*^{\alpha}\left((x-\frac{n}{2^k})^i\chi_{I_{nk}(x)}\right)=\sum_{i=\lceil\alpha\rceil}^{m}\frac{a_{mi}(i)!}{\Gamma(i-\alpha+1)}(x-\frac{n}{2^k})^{i-\alpha}\chi_{I_{nk}(x)}$$

$$+\sum_{i=\lceil\alpha\rceil}^{m}\frac{a_{mi}(i)!}{\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(i-\lceil\alpha\rceil+1)}f_i(x), \tag{15}$$

where

$$a_{mi}=\frac{2^{k(i+\frac{3}{2})}b_{mi}}{\sqrt{\pi}},\quad f_i(x)=\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}\frac{(s-\frac{n}{2^k})^{i-\lceil\alpha\rceil}}{(x-s)^{\alpha-\lceil\alpha\rceil+1}}ds\,\chi_{[\frac{n+1}{2^k},1]}(x).$$

Here, due to the fact that $(x-\frac{n}{2^k})^{i-\alpha}\chi_{I_{nk}(x)}$ is zero outside the interval $[\frac{n}{2^k},\frac{n+1}{2^k}]$, then, the second kind Chebyshev wavelets expansion of this function has only components of basis Chebyshev wavelets $\Psi(x)$ that are non-zero on this interval which yields:

$$(x-\frac{n}{2^k})^{i-\alpha}\chi_{I_{nk}(x)}=\sum_{j=0}^{M-1}e_{ij}\psi_{nj}(x),\quad i=\lceil\alpha\rceil,\lceil\alpha\rceil+1,\ldots,m, \tag{16}$$

where

$$e_{ij}=\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}(x-\frac{n}{2^k})^{i-\alpha}\psi_{nj}(x)w_n^*(x)dx=\sum_{r=0}^{j}\frac{2^{\frac{k+2}{2}}b_{jr}}{\sqrt{\pi}\,2^{k(i-\alpha)}}\int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}}(2^kx-n)^{r+i-\alpha}w_n^*(x)dx$$

$$=\sum_{r=0}^{j}\frac{2b_{jr}}{2^{k(i-\alpha+\frac{1}{2})}}\frac{\Gamma(i+r-\alpha+\frac{3}{2})}{\Gamma(i+r-\alpha+3)}, \tag{17}$$

and $w_n^*(x)=\sqrt{1-\left(2(2^kx-n)-1\right)^2}$.

Also, expanding $f_i(x)$ by the second kind Chebyshev wavelets in each of the intervals $\chi_{I_{lk}(x)}, l=n+1,\ldots,2^k-1$, yields:

$$f_i(x)=\sum_{j=0}^{M-1}\hat{e}_{ij}\psi_{lj}(x), \tag{18}$$

where

$$\hat{e}_{ij}=\int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}}f_i(x)\psi_{lj}(x)w_l^*(x)dx=\sum_{r=0}^{j}\frac{2^{\frac{k+2}{2}}b_{jr}}{\sqrt{\pi}}\int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}}f_i(x)(2^kx-l)^rw_l^*(x)dx. \tag{19}$$

Now by substituting (16)-(19) into (15), we obtain:

$$D_*^{\alpha}\psi_{nm}(x)=\sum_{j=0}^{M-1}\Omega_{\alpha}^{(n)}(m,j)\psi_{nj}(x)+\sum_{l=n+1}^{2^k-1}\sum_{j=0}^{M-1}\hat{\Omega}_{\alpha}^{(n)}(m,j)\psi_{lj}(x), \tag{20}$$

where $\Omega_{\alpha}^{(n)}(m,j)=\sum_{i=\lceil\alpha\rceil}^{m}\Theta_{mji}$ and $\hat{\Omega}_{\alpha}^{(n)}(m,j)=\sum_{i=\lceil\alpha\rceil}^{m}\hat{\Theta}_{mji}$, and

$$\Theta_{mji}=\frac{a_{mi}(i)!}{\Gamma(i-\alpha+1)}\times\sum_{r=0}^{j}\frac{2b_{jr}}{2^{k(i-\alpha+\frac{1}{2})}}\frac{\Gamma(i+r-\alpha+\frac{3}{2})}{\Gamma(i+r-\alpha+3)},$$

and

$$\hat{\Theta}_{mji}=\frac{a_{mi}(i)!}{\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(i-\lceil\alpha\rceil+1)}\times\sum_{r=0}^{j}\frac{2^{\frac{k+1}{2}}b_{jr}}{\sqrt{\pi}}\int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}}f_i(x)(2^kx-l)^rw_l^*(x)dx.$$

After some simplification, Θ_{mji} and $\hat{\Theta}_{mji}$ can be expressed in the following form:

$$\Theta_{mji}=\frac{2^{k\alpha+2(i+1)}(-1)^{m+j-i}(j+1)(i)!(m+i+1)!\Gamma(j-i+\alpha)\Gamma(i-\alpha+\frac{3}{2})}{\sqrt{\pi}(m-i)!(2i+1)!\Gamma(\alpha-i)\Gamma(i-\alpha+1)\Gamma(j+i-\alpha+3)},m\geq\lceil\alpha\rceil,$$

and

$$\hat{\Theta}_{mji}=\frac{2^{k(i+1)+2}b_{mi}(i)!}{\pi\Gamma(\lceil\alpha\rceil-\alpha)\Gamma(i-\lceil\alpha\rceil+1)}\times\sum_{r=0}^{j}b_{jr}\int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}}f_i(x)(2^kx-l)^rw_l^*(x)dx,m\geq\lceil\alpha\rceil.$$

Therefore (20) can be written as:

$$D_*^{\alpha}\psi_{nm}(x)=[\Omega_{\alpha}^{(n)}(m,0),\Omega_{\alpha}^{(n)}(m,1),\ldots,\Omega_{\alpha}^{(n)}(m,M-1)]\Psi_n(x)$$

$$+\sum_{l=n+1}^{2^k-1}[\hat{\Omega}_{\alpha}^{(n)}(m,0),\hat{\Omega}_{\alpha}^{(n)}(m,1),\ldots,\hat{\Omega}_{\alpha}^{(n)}(m,M-1)]\Psi_l(x), \tag{21}$$

where

$$\Psi_s(x)=[\psi_{s0}(x),\psi_{s1}(x),\ldots,\psi_{s(M-1)}(x)],\quad s=0,1,\ldots,2^k-1.$$

This completes the proof.

Remark 2 *For* $\alpha=\lceil\alpha\rceil$*, from Caputo's derivative, we have:*

$$D_*^{\alpha}\psi_{nm}(x)=\begin{cases}0, & 0\le m<\lceil\alpha\rceil,\\ \sum_{j=0}^{M-1}\Omega_{\alpha}^{(n)}(m,j)\psi_{nj}(x), & m\ge\lceil\alpha\rceil.\end{cases}$$

Theorem 4-2 *Let* $\Psi(x)$ *be the second kind Chebyshev wavelets vector defined in (12) and* $\alpha>0(\lceil\alpha\rceil-1<\alpha<\lceil\alpha\rceil)$*, be a positive constant. Then we have:*

$$D_*^{\alpha}\Psi(x)=D^{\alpha}\Psi(x), \tag{22}$$

where D^{α} is the $\hat{m}\times\hat{m}$ operational matrix of fractional derivative of order α of the second kind Chebyshev wavelet and is defined as follows:

$$D^{\alpha}=\begin{pmatrix} B^{\alpha} & F^{\alpha} & \ldots & F^{\alpha} & F^{\alpha}\\ 0 & B^{\alpha} & \ldots & F^{\alpha} & F^{\alpha}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \ldots & B^{\alpha} & F^{\alpha}\\ 0 & 0 & \ldots & 0 & B^{\alpha}\end{pmatrix}, \tag{23}$$

where B^{α} and F^{α} are $M\times M$ matrices given by:

$$B^{\alpha}=\begin{pmatrix} 0 & 0 & \ldots & 0\\ \vdots & \vdots & \ldots & \vdots\\ 0 & 0 & \ldots & 0\\ \Omega_{\alpha}^{(n)}(\xi,0) & \Omega_{\alpha}^{(n)}(\xi,1) & \ldots & \Omega_{\alpha}^{(n)}(\xi,M-1)\\ \Omega_{\alpha}^{(n)}(\xi+1,0) & \Omega_{\alpha}^{(n)}(\xi+1,1) & \ldots & \Omega_{\alpha}^{(n)}(\xi+1,M-1)\\ \vdots & \vdots & \ldots & \vdots\\ \Omega_{\alpha}^{(n)}(M-1,0) & \Omega_{\alpha}^{(n)}(M-1,1) & \ldots & \Omega_{\alpha}^{(n)}(M-1,M-1)\end{pmatrix}, \tag{24}$$

and

$$F^{\alpha}=\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \hat{\Omega}_{\alpha}^{(n)}(\xi,0) & \hat{\Omega}_{\alpha}^{(n)}(\xi,1) & \dots & \hat{\Omega}_{\alpha}^{(n)}(\xi,M-1) \\ \hat{\Omega}_{\alpha}^{(n)}(\xi+1,0) & \hat{\Omega}_{\alpha}^{(n)}(\xi+1,1) & \dots & \hat{\Omega}_{\alpha}^{(n)}(\xi+1,M-1) \\ \vdots & \vdots & \dots & \vdots \\ \hat{\Omega}_{\alpha}^{(n)}(M-1,0) & \hat{\Omega}_{\alpha}^{(n)}(M-1,1) & \dots & \hat{\Omega}_{\alpha}^{(n)}(M-1,M-1) \end{pmatrix}, \tag{25}$$

and $\xi=\lceil \alpha \rceil$.

Proof. It is an immediate consequence of the lemma 4.1.

Remark 3 *From remark 2, it must be noted that for* $\alpha=\lceil \alpha \rceil$, *we have* $F^{\alpha}=0$.

5. Description of the proposed method

In this section, we apply the operational matrix of fractional derivative for second kind Chebyshev wavelet for solving fractional telegraph equation (1) with the boundary conditions:

$$\begin{aligned} u(x,0) &= f_0(x), \quad u(0,t)=g_0(t), \\ u(x,1) &= f_1(x), \quad u(1,t)=g_1(t). \end{aligned} \tag{26}$$

For this purpose, we suppose:

$$u(x,t)=\Psi(x)^T U \Psi(t), \tag{27}$$

where $U=[u_{i,j}]_{\hat{m}\times\hat{m}}$ is an unknown matrix which should be found and $\Psi(.)$ is the vector which is defined in (12). Now using (13) and (22), we obtain:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)=\Psi(x)^T U D^{\alpha}\Psi(t), \tag{28}$$

$$\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}u(x,t)=\Psi(x)^T U D^{\alpha-1}\Psi(t), \tag{29}$$

and

$$\frac{\partial^{2}}{\partial x^{2}}u(x,t)=(D^2\Psi(x))^T U \Psi(t). \tag{30}$$

Also using (13), the function $f(x,t)$ in (1) can be approximated as:

$$f(x,t);\Psi(x)^T B \Psi(t), \tag{31}$$

where $B=[B_{ij}]$ is a $\hat{m}\times\hat{m}$ known matrix with entries $B_{ij}=\big(\psi_i(x),\big(f(x,t),\psi_j(t)\big)\big)$. Substituting (27)-(31) in (1) consequent:

$$\Psi(x)^T\left[U(D^{\alpha}+D^{\alpha-1})+U-(D^2)^T U-B\right]\Psi(t)=0. \tag{32}$$

The entries of vectors $\Psi(x)$ and $\Psi(t)$ in (32) are independent, so we have:

$$H=U(D^{\alpha}+D^{\alpha-1})+U-(D^2)^T U-B=0. \tag{33}$$

Here, we choose $(\hat{m}-2)^2$ equations of (33) as:

$$H_{ij}=0, \quad i,j=2,\dots,\hat{m}-1. \tag{34}$$

We can also approximate the functions $f_0(x)$, $f_1(x)$, $g_0(t)$ and $g_1(t)$ as:

$$f_0(x) = C_1^T \Psi(x), \qquad g_0(t) = C_3^T \Psi(t),$$
$$f_1(x) = C_2^T \Psi(x), \qquad g_1(t) = C_4^T \Psi(t), \tag{35}$$

where C_1, C_2, C_3 and C_4 are known vectors of dimension $\hat{m}$.
Applying (27) and (35) in the boundary conditions (26), we have:

$$\Psi(x)^T U \Psi(0) = \Psi(x)^T C_1, \qquad \Psi(0)^T U \Psi(y) = C_3^T \Psi(t),$$
$$\Psi(x)^T U \Psi(1) = \Psi(x)^T C_2, \qquad \Psi(1)^T U \Psi(y) = C_4^T \Psi(t). \tag{36}$$

The entries of vectors $\Psi(x)$ and $\Psi(t)$ are independent, so from (36) we obtain:

$$\Lambda_1 = U\Psi(0) - C_1 = 0, \qquad \Lambda_3 = \Psi(0)^T U - C_3^T = 0,$$
$$\Lambda_2 = U\Psi(1) - C_2 = 0, \qquad \Lambda_4 = \Psi(1)^T U - C_4^T = 0. \tag{37}$$

By choosing the $\hat{m}$ equations of $\Lambda_j = 0\,(j = 1,2)$ and $\hat{m} - 2$ equations of $\Lambda_j = 0\,(j = 3,4)$, we get $4\hat{m} - 4$ equations, i.e.

$$\Lambda_{ji} = 0, \quad j = 1,2, \quad i = 1,2,\ldots,\hat{m},$$
$$\Lambda_{ji} = 0, \quad j = 3,4, \quad i = 2,3,\ldots,\hat{m}-1. \tag{38}$$

Equetion (34) together (38) give $\hat{m}^2$ equations, which can be solved for u_{ij}, $i, j = 1,2\ldots,\hat{m}$. So the unknown function $u(x,t)$ can be found.

6 . Numerical examples

In this section, we demonstrate the efficiency of the proposed method for numerical solution of the telegraph equation in the form of (1) with the boundary conditions (26).

Example 1 *Consider the time-fractional telegraph equation (1) with* $f(x,t) = x^2 + t - 1$ *and the boundary conditions as:*

$$u(x,0) = x^2, \qquad u(0,t) = t,$$
$$u(x,1) = 1 + x^2, \quad u(1,t) = 1 + t.$$

The exact solution of this problem for $\alpha = 2$ is $u(x,t) = x^2 + t$. Numerical solutions for some different values of α and $t \in [0,1]$ for $\hat{m} = 6(k = 1, M = 3)$ are shown in Fig. 1. The values of exact solution ($\alpha = 2$) and approximate solutions for some different values of α and some nodes (x, y) in $[0,1] \times [0,1]$, for $\hat{m} = 6$ are shown in Table 1.

Table1. Comparison between the exact ($\alpha = 2$) and numerical solutions for Example ??.

(x_i, y_i)	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	**Exact solution**
(0.2,0.2)	0.26551095384623	0.25900735191924	0.25037754779349	0.24000000000000
(0.4,0.4)	0.61739964615400	0.60276654181831	0.58334948253535	0.56000000000000
(0.6,0.6)	1.01739964615401	1.00276654181831	0.98334948253536	0.96000000000000
(0.8,0.8)	1.46551095384624	1.45900735191925	1.45037754779350	1.44000000000000
(1.0,1.0	1.99999999999999	1.99999999999999	1.99999999999999	2.00000000000000

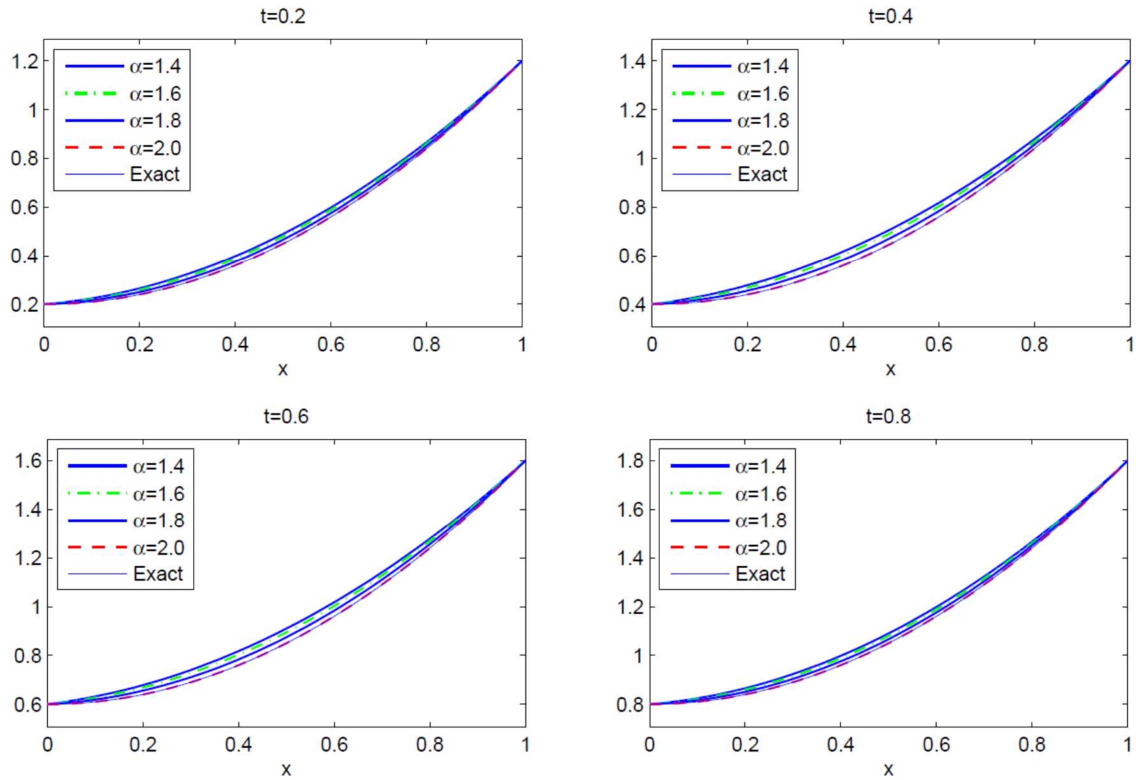

Fig.1. Numerical solutions of Example 1 for some different values of α .

Example 2 *Consider time-fractional telegraph equation (1) with* $f(x,t)=0$ *and the boundary conditions as:*

$$u(x,0)=e^{x}, \quad u(0,t)=e^{-t},$$
$$u(x,1)=e^{x-1}, \quad u(1,t)=e^{1-t}.$$

The exact solution of this problem for $\alpha=2$ is $u(x,t)=e^{x-t}$. Numerical solutions for some different values of α and $t\in[0,1]$ for $\hat{m}=8(k=1,M=4)$ are shown in Fig. 2. The values of the exact solution ($\alpha=2$) and approximate solutions for some different values of α and some nodes (x,y) in $[0,1]\times[0,1]$, for $\hat{m}=8$ are shown in Table 2.

Table 2. Comparison between the exact ($\alpha=2$) and numerical solutions for Example 2.

(x_i,y_i)	$\alpha=1.4$	$\alpha=1.6$	$\alpha=1.8$	**Exact solution**
(0.2,0.2)	1.00325555517139	1.00391875185692	1.00627606867049	1.00000000000000
(0.4,0.4)	1.00541907760911	1.00685195815738	1.01150179716283	1.00000000000000
(0.6,0.6)	1.00546382142790	1.00673580015166	1.01056465711498	1.00000000000000
(0.8,0.8)	1.00334917683010	1.00379783774964	1.00506051184037	1.00000000000000
(1.0,1.0	0.99944928830814	0.99944928830812	0.99944928830815	1.00000000000000

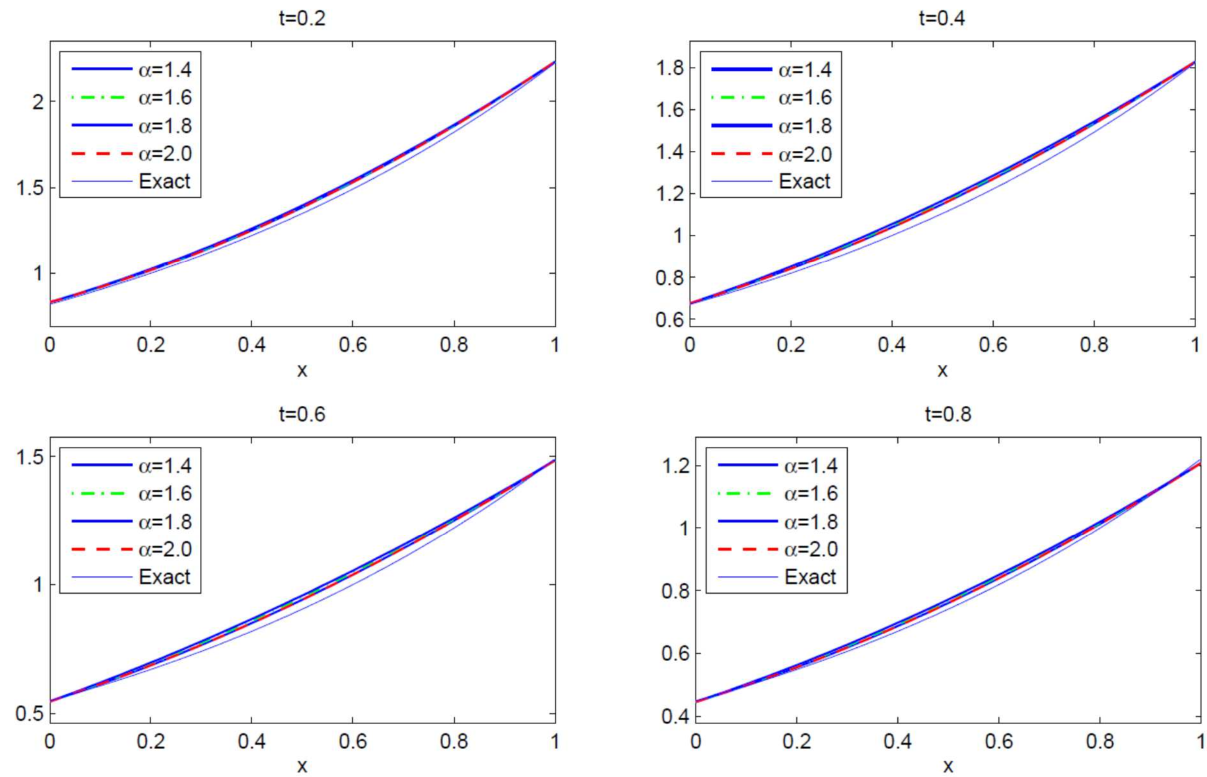

Fig.2. Numerical solutions of Example 2 for some different values of α .

Conclusion

In this paper, a numerical method for approximating the solution of the time-fractional telegraph equation with Dirichlet boundary condition by combining second kind Chebyshev wavelet function with their operational matrix of fractional derivatives was presented. The method was shown that is very convenient for solving boundary value problems. Also, the implementation of the proposed method is very simple and is very efficient for solution of the telegraph problem. Moreover, the proposed method can be used for numerical solution of other kinds of fractional partial differential equations such as Poisson and diffusion equations.

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