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ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS USING DECOMPOSITION METHOD

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ABSTRACT

In this paper, we suggest and analyze a new three-step method for solving nonlinear equations using the decomposition technique which is mainly due to Noor et al. [5]. We show that this new iterative method has fourth-order of convergence. To demonstrate the efficiency and performance of the new method, we tested several numerical examples and results are shown in the table-1.

KEYWORDS: Iterative methods, Two-step methods, Three-step methods, Nonlinear equations, Order of convergence

1. INTRODUCTION

In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations, see [2-6] and the references therein. These methods can be classified as one-step, two-step and three-step methods. Two-step methods have been suggested by combining the well-known Newton method with other one-step implicit methods. In [2] Chun has proposed and studied several one-step and two-step iterative methods with higher-order convergence by using the decomposition technique of Adomian [1]. In the methods of Chun [2], higher-order differential derivatives are involved, which is a serious drawback. To overcome this draw back, we suggest and analyze a family of multi-step methods for solving nonlinear equations using a different type of decomposition, which does not involve the high-order differentials of the function. This new decomposition is mainly due to Noor et al. [5].

2. ITERATIVE METHODS

Consider the nonlinear equation

$$f(x) = 0. \tag{2.1}$$

We assume that α is a simple root of (2.1) and ξ is an initial guess sufficiently close to α . We can rewrite the nonlinear equation (2.1) as a coupled system:

$$f(\gamma) + (x-\gamma)\frac{f'(\gamma)+f'(x)}{2} + g(x) = 0, \tag{2.2}$$

$$g(x) = f(x) - f(\gamma) - (x-\gamma)\frac{f'(\gamma)+f'(x)}{2}, \tag{2.3}$$

where γ is the initial approximation for a zero of (2.1) .

We can rewrite (2.2) in the following form:

$$x = \gamma - \frac{2f(\gamma)}{f'(\gamma)+f'(x)} - \frac{2g(x)}{f'(\gamma)+f'(x)}. \tag{2.4}$$

We can also rewrite (2.4) as:

$$x = c + N(x), \tag{2.5}$$

where

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$$c = \gamma - \frac{2f(\gamma)}{f'(\gamma) + f'(x)}, \tag{2.6}$$

and

$$N(x) = -\frac{2g(x)}{f'(\gamma) + f'(x)}. \tag{2.7}$$

Here $N(x)$ is a linear operator.

We note that if x_0 is the initial guess, then from (2.2) and (2.3), we have

$$f(x_0) = g(x_0). \tag{2.8}$$

As in [6], the solution of (2.5) has the series form,

$$x = \sum_{i=0}^{\infty} x_{i.} \tag{2.9}$$

The nonlinear operator $N(x)$ can be decomposed as

$$N(\sum_{i=0}^{\infty} x_i) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j \right) - N\left(\sum_{j=0}^{i-1} x_j \right) \right\}. \tag{2.10}$$

Combining (2.4), (2.7) and (2.10), we have

$$x = \sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} x_j \right) - N\left(\sum_{j=0}^{i-1} x_j \right) \right\}. \tag{2.11}$$

Thus we have the following iterative scheme:

$$\begin{aligned} x_0 &= c, \\ x_1 &= N(x_0), \\ x_2 &= N(x_0 + x_1) - N(x_0), \\ &\vdots \\ x_{n+1} &= N(x_0 + x_1 + \ldots + x_n) - N(x_0 + x_1 + \ldots + x_{n-1}),\ n = 1, 2, \ldots. \end{aligned} \tag{2.12}$$

Then

$$x_1 + x_2 + \ldots + x_{n+1} = N(x_0 + x_1 + \ldots + x_n),\ n = 1, 2, \ldots,$$

and

$$x = c + \sum_{i=1}^{\infty} x_i. \tag{2.13}$$

It can be shown that the series $\sum_{i=0}^{\infty} x_i$ converges absolutely and uniformly to a unique solution of equation (2.5).

From (2.6), (2.9), (2.10), (2.11) and (2.12), we have

$$x_0 = c = \gamma - \frac{2f(\gamma)}{f'(\gamma) + f'(x)}, \tag{2.14}$$

and

$$x_1 = N(x_0) = -\frac{2g(x_0)}{f'(\gamma) + f'(x_0)} = -\frac{2f(x_0)}{f'(\gamma) + f'(x_0)}. \tag{2.15}$$

This enables us to suggest the following method for solving the nonlinear equation (2.1).

Algorithm *For the give* x_o *compute the approximate solution* x_{n+1} *by the iterative schemes:*

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \; n = 0,1,2,\cdots,$$
$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n)}, \tag{2.16}$$

which is known as Weerakoon and Fernando method [7] and is cubically convergent.

Again by using (2.12), (2.13), (2.14) and (2.15), we conclude that

$$\begin{aligned} x &= c + x_1 \\ &= x_0 + N(x_0) \\ &= \gamma - \frac{2f(\gamma)}{f'(\gamma) + f'(x)} - \frac{2f(x_0)}{f'(\gamma) + f'(x_0)}. \end{aligned} \tag{2.17}$$

Using (2.17), we can suggest the following three-step iterative method for solving nonlinear equation (2.1) as:

Algorithm *For the given* x_o *compute the approximate solution* x_{n+1} *by the iterative schemes (AAU):*

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0,$$
$$y_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n)},$$
$$x_{n+1} = y_n - \frac{2f(y_n)}{f'(x_n) + f'(y_n)}.$$

Again, using (2.8) and (2.12), we can calculate

$$x_2 = N(x_0 + x_1) = -\frac{2f(x_0 + x_1)}{f'(\gamma) + f'(x_0 + x_1)}. \tag{2.18}$$

From (2.11), (2.12), (2.13), (2.14), (2.15) and (2.18), we get

$$\begin{aligned} x &= c + x_1 + x_2 \\ &= x_0 + N(x_0) + N(x_0 + x_1) - N(x_0) \\ &= \gamma - \frac{2f(\gamma)}{f'(\gamma) + f'(x)} - \frac{2f(x_0 + x_1)}{f'(\gamma) + f'(x_0 + x_1)}. \end{aligned}$$

Using this, we can suggest and analyze the following four-step iterative method for solving nonlinear equation (2.1).

Algorithm: *For a given* x_0, *compute the approximate solution by the iterative schemes:*

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0,$$
$$y_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n)},$$
$$s_n = -\frac{2f(y_n)}{f'(x_n) + f'(y_n)},$$
$$x_{n+1} = y_n - \frac{2f(y_n + s_n)}{f'(x_n) + f'(y_n + s_n)}.$$

3. ANALYSIS OF CONVERGENCE

Theorem: *Assume that the function* $f : D \subset \mathrm{R} \to \mathrm{R}$ *for an open interval* D *has a simple root* $\alpha \in D$ *. Let* $f(x)$ *be sufficiently smooth in the neighborhood of the root* α *, then the order of convergence of the method defined by Algorithm 2 is four.*

Proof: The iterative scheme is given by

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)},\ f'(x_n) \neq 0, \tag{3.1}$$

$$y_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n)}, \tag{3.2}$$

$$x_{n+1} = y_n - \frac{2f(y_n)}{f'(x_n) + f'(y_n)}. \tag{3.3}$$

Let α be a simple zero of f . By Taylor's expansion, we have,

$$f(x_n) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O\left(e_n^7\right)\right], \tag{3.4}$$

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O(e_n^6)\right], \tag{3.5}$$

where

$$c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)},\ k = 2, 3, \ldots, \text{ and } e_n = x_n - \alpha. \tag{3.6}$$

From (3.4) and (3.5) we have

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 \\ &\quad + (-4c_5 + 10c_2c_4 - 20c_3c_2^2 + 8c_2^4 + 6c_3^2)e_n^5 + O(e_n^6).\end{aligned} \tag{3.7}$$

Using (3.1) and (3.7) we get

$$z_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + O(e_n^5). \tag{3.8}$$

By Taylor's series, we have

$$f(z_n) = f'(\alpha)\left[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5)\right], \tag{3.9}$$

and

$$\begin{aligned}f'(z_n) = f'(\alpha)[&1 + 2c_2^2 e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^4)e_n^4 \\ &+ (8c_2c_5 - 20c_2^2c_4 + 28c_3c_2^3 - 16c_2^5)e_n^5 + O(e_n^6)].\end{aligned} \tag{3.10}$$

From (3.2), (3.5) and (3.10) we get

$$\begin{aligned}y_n = \alpha &+ (\frac{1}{2}c_3 + c_2^2)e_n^3 + (c_4 + \frac{3}{2}c_2c_3 - 3c_2^3)e_n^4 + (\frac{3}{2}c_5 + 2c_2c_4 - 9c_3c_2^2 \\ &+ 6c_2^4 - \frac{3}{4}c_3^2)e_n^5 + O(e_n^6),\end{aligned} \tag{3.11}$$

implies by Taylor's series,

$$\begin{aligned}f(y_n) = f'(\alpha)[&(\frac{1}{2}c_3 + c_2^2)e_n^3 + (c_4 + \frac{3}{2}c_2c_3 - 3c_2^3)e_n^4 + (\frac{3}{2}c_5 + 2c_2c_4 \\ &- 9c_3c_2^2 + 6c_2^4 - \frac{3}{4}c_3^2)e_n^5 + O(e_n^6)],\end{aligned} \tag{3.12}$$

and

$$f'(y_n) = f'(\alpha)[1 + (c_2c_3 + 2c_2^3)e_n^3 + (2c_2c_4 + 3c_3c_2^2 - 6c_2^4)e_n^4 + (3c_2c_5 + 4c_2^2c_4 - 18c_3c_2^3 + 12c_2^5 - \frac{3}{2}c_2c_3^2)e_n^5 + O(e_n^6)]. \tag{3.13}$$

Finally using (3.3), (3.5), (3.12) and (3.13), we obtain

$$x_{n+1} = \alpha + (\frac{1}{2}c_2c_3 + c_2^3)e_n^4 + (\frac{3}{4}c_3^2 + \frac{5}{2}c_3c_2^2 + c_2c_4 - 4c_2^4)e_n^5 + O(e_n^6), \tag{3.14}$$

implies

$$e_{n+1} = (\frac{1}{2}c_2c_3 + c_2^3)e_n^4 + (\frac{3}{4}c_3^2 + \frac{5}{2}c_3c_2^2 + c_2c_4 - 4c_2^4)e_n^5 + O(e_n^6).$$

Hence proved.

4. Numerical Examples

In this section we consider some numerical examples to demonstrate the performance of the new developed iterative method. We compare Chun method [2] (CHU), Noor and Noor method [4] (NR3), Noor and Noor method [3] (NR1) and Noor and Noor method [5] (NR2) with the new developed method (AAU). All the computations for above mentioned methods, are performed using software Maple 9, precision 60 digits and $\varepsilon = 10^{-18}$ as tolerance and also the following criteria is used for estimating the zero:

(i) $\delta = |x_{n+1} - x_n| < \varepsilon.$

(ii) $|f(x_n)| < \varepsilon.$

(iii) Maximum numbers of iterations $= 500.$

We use the following examples for numerical testing and results are given in the Tables of Results.

Example	**Exact Zero**
$f_1 = \sin^2(x) - x^2 + 1,$	$\alpha = 1.4044916482153411$
$f_2 = x^2 - e^x - 3x + 2,$	$\alpha = 0.2575302854398608$
$f_3 = \cos(x) - x,$	$\alpha = 0.7390851332156067$
$f_4 = (x-1)^3 - 1,$	$\alpha = 2.0$
$f_5 = x^3 - 10,$	$\alpha = 2.15443469003188372$
$f_6 = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$	$\alpha = -1.2076478271309191$
$f_7 = e^{x^2+7x-30} - 1,$	$\alpha = 3$

Table - 1

	n	$f(x_n)$	δ
$f_1,\ x_0 = -1$			
NW	7	-1.0e - 50	7.3e - 26
NR1	7	-8.0e - 10	8.9e - 16
NR2	6	-1.5e - 31	3.6e - 16
NR3	8	3.8e - 53	4.2e - 27
CHU	15	-1.6e - 9	1.0e - 28
AAU	4	-1.4e - 96	1.0e - 24
$f_2,\ x_0 = 2$			
NW	6	2.9e - 55	9.1e - 28
NR1	6	7.3e - 50	4.5e - 25
NR2	6	5.1e - 37	1.0e - 18
NR3	6	-1.2e - 29	5.8e - 15
CHU	5	1.5e - 32	2.0e - 16
AAU	4	7.4e - 126	1.5e - 31
$f_3,\ x_0 = 1.7$			
NW	5	-2.0e - 32	2.3e - 16
NR1	5	-2.1e - 33	7.5e - 17
NR2	5	-7.5e - 32	3.5e - 16
NR3	6	6.2e - 55	1.2e - 27
CHU	5	-1.3e - 34	1.9e - 17
AAU	4	0	1.0e - 54
$f_4,\ x_0 = 3.5$			
NW	8	2.0e - 42	8.2e - 22
NR1	7	1.7e - 29	2.4e - 15
NR2	8	2.6e - 54	1.1e - 27
NR3	6	-6.0e - 41	4.4e - 21
CHU	6	1.9e - 28	8.0e - 15
AAU	5	0	1.2e - 39
$f_5,\ x_0 = 1.5$			
NW	7	2.0e-54	5.4e-28
NR1	7	1.3e-31	1.4e-16
NR2	6	3.8e-28	8.0e-15
NR3	7	-1.3e-39	1.4e-20
CHU	27	2.3e-31	1.9e-16
AAU	4	1.0e-103	1.5e-26

$f_6,\ x_0 = -2$			
NW	9	-2.2e-40	2.7e-21
NR1	8	-3.0e-9	1.3e-17
NR2	9	-4.3e-42	3.8e-22
NR3	7	1.8e-37	7.7e-20
CHU	7	-6.0e-9	6.4e-18
AAU	5	-1.3e-61	1.9e-16
$f_7,\ x_0 = 3.5$			
NW	13	1.5e-47	4.2e-25
NR1	11	5.0e-33	7.6e-18
NR2	13	1.0e-50	1.1e-26
NR3	8	-4.6e-37	7.3e-20
CHU	9	4.8e-31	7.5e-17
AAU	7	1.3e-59	2.2e-16

5. CONCLUSION

In the Table-1, we observe that our iterative method (AAU) is comparable with all the cited methods and gives better results. The technique and idea of this paper can be developed to higher-order multi-step iterative methods for solving nonlinear equations, as well as a system of nonlinear equations.

REFERENCES

1) R. G. Adomian, Nonlinear stochastic systems and application to physics, Kluwer Academic publishers, *Dodrecht, 1989*.
2) C. Chun, Iterative methods improving Newton's method by the decomposition method, Comp. Math. Appl., *50 (2005), 1559-1568*.
3) M. A. Noor and K. I. Noor, Three step iterative methods for nonlinear equations, Appl. Math. Comput., *183(1), (2006), 322-327*.
4) M. A. Noor and K. I. Noor, Some iterative schemes for nonlinear equations, Appl. Math. Comput., *183(2), (2006), 774-779*.
5) M. A. Noor, K. I. Noor, S. T. Mohyud-Din and A. Shabbir, An iterative method with cubic convergence for nonlinear equations, App. Math. Comput. *183(2), (2006), 1249-1255*.
6) K. I. Noor and M. A. Noor, Iterative methods with fourth-order convergence for nonlinear equations, Appl. Math. Comput., *189(1), (2007), 221-227*.