Flow of Oldroyd-B Fluid between Two Inclined and Oscillating Plates

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Received: June 9, 2015
Accepted: September 21, 2015

ABSTRACT
The studies of this paper related to the unsteady magnetohydrodynamics (MHD) incompressible Oldroyd-B fluid between two periodically vibrated inclined plates. The basic governing equations of the above mentioned fluid are modeled and simplified in the form of nonlinear PDEs. The solution of the above non-linear PDEs has been obtained by using two analytical methods, namely HPM and OHAM. The comparison of these two methods are also analyzed and found in very good agreement. Finally, the physical effects of modelled parameters have been studied graphically.

KEYWORDS: Unsteady motion, thin film, MHD, Oldroyd-B fluid, inclined belt, OHAM and HPM.

I. INTRODUCTION

In everyday life and in engineering flow of non-Newtonian fluid are frequently occurred. It is ubiquitous in nature and technologies. Therefore, to understand there mechanics is essential in most of applications. Most of it problems appeared in several areas varying from blood to the engineering and technologies problem. In non-Newtonian fluids Thin film flows have a large variety of practical applications in nonlinear sciences and engineering industries. For modelling of non-Newtonian fluid flow problem we used several model lik (second grade, third grad, fourth grad oldroyd four, oldroyed-B) etc. fluids model by which we got linear or nonlinear ode,s or pde,s which may be solved exactly, analytically or numerically. For solution of these problems different varieties of methods are used, in which (ADM, VAM, HAM, HPM, OHAM) are frequently used. In this work we have modelled a non linear pde,s by using oldroyd-B fluid model. Solution of the problem is obtained by using OHAM and HPM. Anakira et all [1] have used OHAM for the solution of Delay differential equations. Aksel et all [2] have been examined the solutions of for some unsteady of oldroyd-B fluids in unidirectional flows. Ayub et all [3] discussed the oscillating motions in plane wall for said fluid. Burdijan [4] take it in particular class and give a brief discussion on it. Fetecau et al. [5-6] studied oldroyd –B fluid in form of constantly accelerating flow over a flat plate and in second paper he discussed transient oscillating motion using same model in cylindrical domains. In both case he obtained the exact solutions.

The related work can be seen in [3-6].Ghosh et all [7-9] examined Oldroyd-B fluid under the effect of MHD in different case that is pulsating flow, Chanel flow and induced by rectified sine pulses. All of details is given briefly in their works. Gul et al. [10, 12] investigated the MHD thin film flow of non-Newtonian fluid on a vertical and in inclined oscillating belts. They obtained analytical solution of using the Asymptotic perturbation methods (ADM, OHAM and HPM). The obtained results of the lift and drainage problems for velocity and as well as for temperature field are compared and given graphically. The properties of several physical parameters are also explained. Hameedat all [13] discussed thin film fluids in vertical belt. Haitao [14] et all studied unidirectional flow of an oldroyd-B fluid. He JH [15-20] has been given a brief disruption on HPM methods. Khan et al [21-23]studied Oldroyed –B fluid in rotating system with effect of MHD and also studied it in unsteady stat and find some exact solutions for it. They examined it in porous space. Kashkari [24] examined the OHAM application and approximate solution of nonlinear Kawahara equation. Lie et al [25-26] studied Axial MHD flow of generalized Oldroyd-B fluid due to two oscillating cylinder. They gives basic introduction of HPM with detail examples. Marincu et al. [26-29] applied OHAM on non-linear steady flow problem modelled from fourth order fluid and find approximate solution. They used OHAM for thin film flow problem and for nonlinear oscillators with discontinuities. Mabood et al [30-31] gives application of Optimal homotopy asymptotic method for the approximate solution and they also examined the heat transfer on MHD stagnation point flow in porous medium. Nofel [32] examined temperature conduction and fractional vanderpol damped nonlinear oscillator.

Shahid et al [33] studied unsteady flow of Oldroyd-B fluid in steady states. Transient solution have been obtained with used of Laplace and Fourier series. Shah et al. [34] examined thin film flow of third order fluid on moving inclined plane and found it solution by using OHAM. Siddiqui et al. [35-37] examined the thin film
flow of a third order fluid over an inclined plane as well they studied it in moving vertical belt. For solution of the non-linear problems they used HPM in first case and OHAM in other case.

II. Basic Equation

Momentum and continuity equation are written as

\[ \nabla \cdot \mathbf{v} = 0, \]

\[ \rho \frac{D\mathbf{v}}{Dt} = \nabla \mathbf{T} + \mathbf{J} \times \mathbf{B} + \rho g \sin \theta. \]

Where

\[ \mathbf{J} \times \mathbf{B} = \begin{bmatrix} 0, -\sigma B_0^2 u, 0 \end{bmatrix}, \]

\[ \mathbf{J} = \sigma (\mathbf{E} + \mathbf{U} \times \mathbf{B}), \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \]

\( \frac{D}{Dt} \) shows the material time derivative, \( \rho \) is density, and \( g \) is body force and \( \mathbf{v} \) is the velocity. Where \( \mathbf{B} \) is the uniform induction field, \( \sigma \) is the electrical conductivity and \( \mathbf{J} \) is current density.

The above model can be reduced to different types of fluid depend on \( \lambda_1 \) (relaxation time) and \( \lambda_2 \) (retardation time). In equation (5), if \( \lambda_1 = \lambda_2 \), the fluid becomes viscous. When \( \lambda_1 = 0 \), it becomes a Maxwell fluid and reduced to Oldroyd-B fluid when \( 0 < \lambda_1 < \lambda_2 < 1 \).

The cauchy stress tensor, \( \mathbf{T} \) is

\[ \mathbf{T} = -\mathbf{P} \mathbf{I} + \mathbf{\tau}, \]

\( \mathbf{P} \) is isotropic stress.

\[ \mathbf{T} + \lambda_1 \frac{D\mathbf{T}}{Dt} = \mu \left[ 1 + \lambda_2 \frac{D}{Dt} \right] \mathbf{A}_1, \]

\( \mathbf{T} \) is used for extra stress tensor and \( \mu \) is used for coefficient of viscosity.

\[ \mathbf{A}_0 = \mathbf{I}, \quad \mathbf{A}_1 = \nabla \mathbf{v} + (\nabla \mathbf{v})^T, = \nabla \mathbf{v}, \]

\( \mathbf{A}_1 \) is the Rivlin Ericksen stress tensor.

III. Formulation of the Problem

Consider incompressible non-Newtonian Oldroyd-B fluid between two inclined and parallel plates \( y = h \) and \( y = -h \). Both plates are oscillating. The configuration of fluid flow is along the y-axis and perpendicular to x-axis. A transvers uniform magnetic field is applied on inclined direction where the plate is non-conducting. The pressure \( \mathbf{P} \) is kept constant, that is pressure gradient is zero. The flow is supposed to be unsteady flow, laminar. Gravitational force and magnetic force causes the fluid motion.
subject to the boundary conditions

\[ u(-h,t) = U \cos \omega t, \quad u(h,t) = U \cos \omega t \]  

(10)

Where $\omega$ is the frequency of the oscillating belt.

The momentum equation (2) is reduced to

\[ \rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin \theta - \sigma B_0^2 v, \]  

(11)

\[ \frac{\partial p}{\partial y} = \frac{\partial \tau_{xy}}{\partial y}, \]  

(12)

\[ \frac{\partial p}{\partial z} = 0. \]  

(13)

It follows from (7) and (9) that

\[ \tau_{xx} + \lambda_1 \left[ \frac{\partial \tau_{xx}}{\partial t} - 2 \tau_{xy} \frac{\partial v}{\partial y} \right] = -2 \mu \lambda_2 \left[ \frac{\partial v}{\partial y} \right]^2, \]  

(14)

\[ \tau_{yy} + \lambda_1 \left[ \frac{\partial \tau_{xy}}{\partial t} - \tau_{xy} \frac{\partial v}{\partial y} \right] = \mu \left( \frac{\partial v}{\partial y} \right) + \lambda_2 \mu \left( \frac{\partial^3 v}{\partial t^2 \partial y} \right), \]  

(15)

\[ \tau_{yy} + \lambda_1 \frac{\partial \tau_{xy}}{\partial t} = 0, \]  

(16)

Equation (16) reduces to

\[ \tau_{yy} = \Psi(y) e^{-\lambda t}. \]  

(17)

Where $\Psi(y)$ any arbitrary function. When $t << 0$, then $\tau_{yy} = 0$. Equations (11) and (15) and in the presence of zero pressure gradient, we obtain

\[ \left[ 1 + \lambda_1 \frac{\partial}{\partial t} \right] \frac{\partial v}{\partial t} = v \left[ 1 + \lambda_2 \frac{\partial}{\partial t} \right] \frac{\partial^2 v}{\partial y^2} - \frac{\sigma B_0^2}{\rho} v \left[ 1 + \lambda_1 \frac{\partial}{\partial t} \right] v + \rho g \sin \theta. \]  

(18)

Introducing non-dimensional variables

\[ v = \frac{\tilde{v}}{V}, \quad y = \frac{y}{\delta}, \quad t = \frac{\tilde{t}}{\rho \delta^2}, \quad K_1 = \frac{\lambda_1 \delta \omega^2 \rho}{\mu}, \quad K_2 = \frac{\lambda_2 \delta \omega^2 \rho}{\mu}, \quad M = \frac{\sigma B_0^2 \delta^2}{\mu}. \]  

(19)

where, $M$ is the magnetic parameter, $\omega$ is the oscillating parameter, $K_1$ is the relaxation parameter, $K_2$ is the retardation parameter and $m$ is the gravitational parameter.

Using (19) in (18) and (10) and dropping bars we obtain

\[ \left[ 1 + K_1 \frac{\partial}{\partial t} \right] \frac{\partial u}{\partial t} = \left[ 1 + K_2 \frac{\partial}{\partial t} \right] \frac{\partial^2 u}{\partial y^2} - M \left[ 1 + \lambda_1 \frac{\partial}{\partial t} \right] u + m. \]  

(20)

\[ v(-1,t) = \cos \omega t, \quad v(1,t) = \cos \omega t, \]  

(21)

IV. The HPM Solution for the Problem

By applying HPM to (20) and (21) component form is asT

\[ P^0 : \frac{\partial^2 v_0}{\partial y^2} = -m, \]  

(45)

\[ P^1 : \frac{\partial^2 v_1}{\partial y^2} = 2 \frac{\partial^2 v_0}{\partial y^2} - k_1 \frac{\partial^2 v_0}{\partial t^2} + k_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 v_0}{\partial y^2} \right) - M k_1 \frac{\partial v_0}{\partial t} + 2m - M v_0, \]  

(46)
\[ P^2 : \frac{\partial^2 v_2(y,t)}{\partial y^2} = 2 \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial v_1}{\partial t} - K_y M \frac{\partial^2 v_1}{\partial t^2} + K_y \frac{\partial}{\partial t} \left( \frac{\partial^2 v_1}{\partial y^2} \right) - Mv_1, \]  

(47)

Solutions of equations (22-24) using boundary condition in equation (20) is

\[ v_0(y,t) = \left\{ \frac{1}{2} m (y^2 - 1) + \cos \left[ t \omega \right] \right\}. \]  

(48)

\[ v_1(y,t) = \frac{1}{24} \left\{ \begin{array}{l}
M M(300 + 61 M - 360 y^2 - 75 m M y^2 + 60 y^4 + 15 M y^4 - M y^6) + M \cos \left[ t \omega \right](7200 + 150 M - 720 y^2 - 180 M y^2 + 30 M x^4) + \omega^2 \cos \left[ t \omega \right] c_1 (1 - y^2) - 12 M \omega \sin \left[ t \omega \right] K_1 (1 - y^2) + 12 M \omega \sin \left[ t \omega \right] c_1 (1 - y^2) - 12 M \omega \sin \left[ t \omega \right] K_1 (1 - y^2) + 12 M \omega \sin \left[ t \omega \right] c_1 (1 - y^2) - 12 M \omega \sin \left[ t \omega \right] K_1 (1 - y^2)
\end{array} \right. \]  

(49)

\[ v_2(y,t) = \frac{1}{24} \left\{ \begin{array}{l}
M M(300 + 61 M - 360 y^2 - 75 m M y^2 + 60 y^4 + 15 M y^4 - M y^6) + M \cos \left[ t \omega \right](7200 + 150 M - 720 y^2 - 180 M y^2 + 30 M x^4) + \omega^2 \cos \left[ t \omega \right] c_1 (1 - y^2) - 12 M \omega \sin \left[ t \omega \right] K_1 (1 - y^2) + 12 M \omega \sin \left[ t \omega \right] c_1 (1 - y^2) - 12 M \omega \sin \left[ t \omega \right] K_1 (1 - y^2) + 12 M \omega \sin \left[ t \omega \right] c_1 (1 - y^2) - 12 M \omega \sin \left[ t \omega \right] K_1 (1 - y^2)
\end{array} \right. \]  

(50)

The series solutions of velocity profile up to second component is

\[ v(y,t) = v_0(y,t) + v_1(y,t) + v_2(y,t). \]  

(51)

V. The OHAM Solution for the Problem

The component problems of velocity profile are when OHAM method is applied

\[ P^0 : \frac{\partial^2 v_0(y,t)}{\partial y^2} = -m. \]  

(53)

\[ P^1 : \frac{\partial^2 v_1(y,t)}{\partial y^2} = -K_y c_1 \frac{\partial^2 v_0}{\partial t^2} + (1 + c_1) \frac{\partial^2 v_0}{\partial y^2} + K_y c_1 \frac{\partial}{\partial t} \left( \frac{\partial^2 v_0}{\partial y^2} \right) - c_1 (1 + M K_1) \frac{\partial v_0}{\partial t} + m (1 + c_1) - M c_1 v_0 \]  

(54)
\[ p^2 \frac{\partial^2 v_i}{\partial y^2} = -K_1 c_1 \frac{\partial^2 v_i}{\partial t^2} + c_2 \frac{\partial^2 v_i}{\partial y^2} + k_2 c_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 v_i}{\partial y^2} \right) + \left(1 + c_1 \right) \frac{\partial^2 v_i}{\partial y^2} + k_2 c_1 \frac{\partial}{\partial t} \frac{\partial^2 v_i}{\partial y^2} \]  

\[ (55) \]

\[-c_1 \left(1 + M K_1 \right) \frac{\partial v_i}{\partial t} + c_2 \left( m - M v_0 \right) - M c_1 v_i - c_2 \left(1 + M K_1 \right) \frac{\partial v_0}{\partial t} \]

Solutions to equations (30-32) using boundary condition in equation (21) are

\[ v_0 \left(y,t \right) = \left[ \frac{1}{2} m \left( y^2 - 1 \right) + \cos \left[ t \omega \right] \right] \]  

\[ (56) \]

\[ v_i \left(y,t \right) = \frac{1}{24} m \left[ \left( 5 - 6 y^2 \right) + 12c_1 M \cos \left[ t \omega \right] \left( -1 - y^2 \right) - 12c_1 M \sin \left[ t \omega \right] \left( y^2 - 1 \right) \right] \]  

\[ (57) \]

The second component solution for velocity (in OHAM) is very large, therefore, only graphical representations up to second order are given.

The series solutions of velocity profile is obtained as

\[ v \left(y,t \right) = v_0 \left(y,t \right) + v_1 \left(y,t \right) + v_2 \left(y,t \right) \]  

\[ (58) \]

The values of \( C_i \) for the velocity components are

\[ C_1 = -1.0575680997844723, C_2 = 0.008909892406518072. \]

**Table 1** Numerical Comparison of OHAM and HPM for the velocity profile, when

\[ M = 0.02, t = 5, k_1 = 0.02, k_2 = 0.09, t = 10, \omega = 0.09, m = 4. \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>OHAM</th>
<th>ADM</th>
<th>Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2.63195</td>
<td>2.58258</td>
<td>0.049368</td>
</tr>
<tr>
<td>0.1</td>
<td>2.61188</td>
<td>2.56287</td>
<td>0.049063</td>
</tr>
<tr>
<td>0.2</td>
<td>2.55167</td>
<td>2.50377</td>
<td>0.047905</td>
</tr>
<tr>
<td>0.3</td>
<td>2.45131</td>
<td>2.4053</td>
<td>0.0460163</td>
</tr>
<tr>
<td>0.4</td>
<td>2.31078</td>
<td>2.26752</td>
<td>0.0432602</td>
</tr>
<tr>
<td>0.5</td>
<td>2.13004</td>
<td>2.09052</td>
<td>0.0395247</td>
</tr>
<tr>
<td>0.6</td>
<td>1.90906</td>
<td>1.87439</td>
<td>0.0346659</td>
</tr>
<tr>
<td>0.7</td>
<td>1.64779</td>
<td>1.61928</td>
<td>0.028508</td>
</tr>
<tr>
<td>0.8</td>
<td>1.34616</td>
<td>1.32532</td>
<td>0.0208429</td>
</tr>
<tr>
<td>0.9</td>
<td>1.00413</td>
<td>0.992699</td>
<td>0.0114309</td>
</tr>
<tr>
<td>1.0</td>
<td>0.62161</td>
<td>0.62161</td>
<td>-1.8311 \times 10^{-18}</td>
</tr>
</tbody>
</table>

*Fig.2*, Comparison of OHAM and HPM solutions for velocity profile by when

\( \omega = 0.2, m = 0.4, M = 0.5, t = 5, k_1 = 0.6, k_2 = 0.3. \)
Fig. 3: Influence of different time level when $\omega = 0.2, m = 0.4, M = 0.5, t = 5, k_1 = 0.6, k_2 = 0.3$.

Fig. 4: Influence of magnetic parameter on velocity profile when $\omega = 0.2, m = 0.5, t = 5, k_1 = 0.5, k_2 = 0.7$.

Fig. 5: Velocity distribution at different time level $\omega = 0.2, m = 0.4, t = 1, k_1 = 0.5, k_2 = 0.7$.  

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"Fig. 6", Velocity distribution graphs when \( \omega = 0.2, m = 0.4, M = 0.5, t = 5, k_1 = 0.6, k_2 = 0.3, \)

"Fig. 7", Effect of non-Newtonian parameter \( k_1 \) on velocity profiles when \( m = 0.4, M = 0.3, t = 1, k_2 = 0.7. \)

"Fig. 8", Effect of non-Newtonian parameter \( k_2 \) on velocity profiles when \( \omega = 0.2, y = 0.4, M = 0.5, t = 5, k_1 = 0.6. \)
VI. RESULTS AND DISCUSSION

In this work we have studied Oldroyd-B fluid in two inclined oscillating parallel plates. The fluid is considered in unsteady form under the effect of MHD. The obtained results from both OHAM and HPM methods are compared. We have found that these results are in excellent agreement. Tables [1] show the numerical comparisons of OHAM and HPM. The absolute errors between both methods are calculated and shown. Fig. [1] shows the geometry of the problem. Fig. 2 analyse the comparison of OHAM and HPM solutions by using various values of parameters that is \((\omega, M, t, k_2, k_1)\). Fig. [3, 5, 6] show the influence of velocity distribution using parameters \((\omega, M, t, k_2, k_1)\) and show that velocity field is increases by increasing these parameters. All results for the Oldroyd-B fluid near the plates are showed in the \(y\)-coordinate only for a specific domain \(y \in [0, 1]\). The influence \(M\) on velocity field is shown in Fig. [4] which give a result that if we increase it, the motion of fluid flow is also increased. The effects of \(K_1\) (relaxation time parameter) and \(K_2\) (retardation time parameter) are shown in Figs. [7, 8]. Increase in these parameters increases the velocity profile. Fig. [9] show the gravity effect \(m\). Increasing it fluid motion is clearly increased.

VII. CONCLUSION

We have concluded the excellent agreement of HPM and OHAM for the mentioned modelled problem.

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doi:10.1371/journal.pone.0103843.


doi:10.1371/journal.pone.0097552.


