# Teleparallel Lie Symmetries of FRW Spacetime by Using Diagonal Tetrad 

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#### Abstract

In relativity theory Lie derivative of different quantities (physical or geometrical) define symmetries. The study of symmetries in general relativity not only gives us new solutions of Einstein's field equations but also classify those solutions. In this paper Killing and homothetic symmetries of FRW spacetime are discussed in an equivalent theory of gravitation known as teleparallel theory of gravitation. Our findings show that FRW spacetimes admit only four teleparallel Killing vector fields and do not possess proper teleparallel homothetic vector field. KEYWORDS: Teleparallel Lie derivative, Torsion, Weitzenböck connection, Tetrad field.


## 1. INTRODUCTION

General theory of relativity is one such theory of gravitation which gave momentum to our understanding of universe and enables us to study different aspects of universe through the powerful knowledge provided by this theory. The study of conservation laws through symmetries of the metric for a spacetime is one aspect that helps us to understand and expose the hidden realities of the universe. It is well established fact that symmetries of a spacetime metric give rise to conservation laws [1]. For this reason the study of different symmetries remained an important topic to discuss. The symmetries of spacetime metric like Killing and homothetic vector fields were studied in the presence of curvature in the spacetime [2-5].

The laws of nature described by general relativity seem true and most of them have been proved through experiments. The validity of these laws is recorded at classical level but at quantum level relativity theory do not provide a meaningful description. For this reason, Einstein himself introduced teleparallel theory of gravitation. This theory is based upon Weitzenböck connection [6]. In terms of Weitzenböck connection the spacetime curvature is zero and it has torsion. This torsion is now responsible for matter interaction and it plays the role of a force [7]. Though this theory remained unsuccessful in describing interaction at quantum level but researchers are studying it as an alternative theory of gravitation. The study of symmetries in this new description has been started with the definition of teleparallel Lie derivative and application of that to the Einstein's universe for teleparallel Killing vector fields [8]. After that, number of papers have been published by many authors on teleparallel Killing, homothetic and conformal vector fields [9-16].

Homothetic vector fields which preserve the metric of a spacetime to a constant factor is important in both relativity theory and teleparallel theory as it give one more symmetry generator than Killing vector field. We are therefore, interested in this paper to explore teleparallel Killing and homothetic vector fields for the Lorentzian manifold of FRW $K=+1$ model. Before going to discuss our main results we shall give an introduction of the basic terms involve in this research.

## 2. Teleparallel Theory (Some Basic Terms):

In teleparallel theory a covariant derivative on tensor of rank 2 works as [7]
$\nabla_{\rho} Y_{\mu \nu}=Y_{\mu \nu, \rho}-\Gamma^{\theta}{ }_{\rho \nu} Y_{\mu \theta}-\Gamma^{\theta}{ }_{\mu} Y_{\nu \theta}$.

[^0]here comma stands for partial derivative and $\Gamma^{\theta}{ }_{\rho v}$ represents Weitzenböck connections. They are obtainable from tetrad through relation [7]
\[

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\theta}=B_{a}^{\theta} \partial_{\nu} B^{a} \mu \tag{2}
\end{equation*}
$$

\]

where $B_{a}{ }^{\nu}$ and $B^{a} \mu$ are non-trivial tetrad field and inverse tetrad field respectively. Tetrad fields themselves satisfy the relation

$$
\begin{equation*}
B_{\mu}^{a} B_{a}^{v}=\delta_{\mu}^{v}, \quad B_{\mu}^{a} B_{b}^{\mu}=\delta_{b}^{a} \tag{3}
\end{equation*}
$$

The Riemannian metric can be generated from these tetrad fields as
$g_{\mu \nu}=\eta_{a b} B^{a}{ }_{\mu} B_{\nu}^{b}$.
where $\eta_{a b}=\operatorname{diag}(-1,1,1,1)$ is the Minkowski metric. Weitzenböck connections give rise to torsion tensor as

$$
\begin{equation*}
T_{\mu \nu}^{\theta}=\Gamma^{\theta}{ }_{\nu \mu}-\Gamma^{\theta}{ }_{\mu \nu} \tag{5}
\end{equation*}
$$

This torsion is anti symmetric in its last two indices. Teleparallel Killing equations for the vector filed $X$ can be obtained from the equation [8]
${ }_{X}^{L} g_{\alpha \beta}=g_{\alpha \beta, \rho} X^{\rho}+g_{\rho \beta} X^{\rho}{ }_{, \alpha}+g_{\alpha \rho} X^{\rho}{ }_{, \beta}+X^{\rho}\left(g_{\theta \beta} T^{\theta}{ }_{\alpha \rho}+g_{\alpha \theta} T^{\theta}{ }_{\beta \rho}\right)=0$,
where $L_{X}^{T}$ represents Lie derivative in teleparallel theory. For finding teleparallel proper homothetic vector fields this equation will extend to

$$
\begin{equation*}
{\underset{X}{L}}_{L^{T}} g_{\mu \nu}=2 \alpha g_{\mu \nu}, \quad \alpha \in R \backslash\{0\} \tag{7}
\end{equation*}
$$

## 3. MAIN RESULTS

The line element for FRW $K=+1$ spherical model in its usual coordinates system is given by

$$
\begin{equation*}
d s^{2}=-d t^{2}+\Phi^{2}(t)\left[d \chi^{2}+\operatorname{Sin}^{2} \chi\left(d \theta^{2}+\operatorname{Sin}^{2} \theta d \phi^{2}\right)\right] \tag{8}
\end{equation*}
$$

where $\Phi$ is no-where zero functions of $t$ only. A simple use of relation (4) gives us the following tetrad and inverse tetrad components as

$$
\begin{gather*}
B_{\mu}^{a}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \Phi(t) & 0 & 0 \\
0 & 0 & \Phi(t) \sqrt{\operatorname{Sin} \chi} & 0 \\
0 & 0 & 0 & \Phi(t) \sqrt{\operatorname{Sin} \chi \operatorname{Sin} \theta}
\end{array}\right)  \tag{9}\\
B_{a}^{\mu}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\Phi(t)} & 0 & 0 \\
0 & 0 & \frac{(\operatorname{Sin} \chi)^{-\frac{1}{2}}}{\Phi(t)} & 0 \\
0 & 0 & 0 & \frac{(\operatorname{Sin} \chi \operatorname{Sin} \theta)^{-\frac{1}{2}}}{\Phi(t)}
\end{array}\right)
\end{gather*}
$$

Using relation (5) to get the torsion components as

$$
\begin{align*}
& T_{01}^{1}=-T^{1}{ }_{10}=\frac{\Phi^{\bullet}}{\Phi}, T^{2}{ }_{02}=-T^{2}{ }_{20}=\frac{\Phi^{\bullet}}{\Phi}, T^{3} 03=-T^{3} 30=\frac{\Phi^{\bullet}}{\Phi}  \tag{11}\\
& T^{2}{ }_{12}=-T^{2}{ }_{21}=\frac{1}{2} \cot \chi, T^{3}{ }_{13}=-T^{3}{ }_{31}=\frac{1}{2} \cot \chi, T^{3} 23=-T^{3} 32=\cot \theta
\end{align*}
$$

where dot denotes the derivative with respect to $t$. A vector field $X$ is called a teleparallel homothetic vector field if it satisfies equation (7). Expanding equation (7) with the help of equations (8) and (11), the following ten non linear, coupled partial differential equations are obtained

$$
\begin{gather*}
X_{, 0}^{0}=\alpha, X^{1}, 1=\alpha  \tag{12}\\
X_{, 1}^{0}-\Phi^{2}(t) X_{, 0}^{1}-\Phi(t) \Phi^{\bullet}(t) X^{1}=0  \tag{13}\\
X_{, 2}^{0}-\Phi^{2}(t) \operatorname{Sin}^{2} \chi X^{2}, 0-\Phi(t) \Phi^{\bullet}(t) \operatorname{Sin}^{2} \chi X^{2}=0  \tag{14}\\
X_{, 3}^{0}-\Phi^{2}(t) \operatorname{Sin}^{2} \chi \operatorname{Sin}^{2} \theta X_{, 0}^{3}-\Phi(t) \Phi^{\bullet}(t) \operatorname{Sin}^{2} \chi \operatorname{Sin}^{2} \theta X^{3}=0  \tag{15}\\
X_{, 2}^{1}+\operatorname{Sin}^{2} \chi X_{, 1}^{2}+\frac{1}{2} \operatorname{Sin} \chi \operatorname{Cos} \chi X^{2}=0  \tag{16}\\
\operatorname{Sin}^{2} \chi \operatorname{Sin}^{2} \theta X_{,, 1}^{3}+X_{, 3}^{1}+\frac{1}{2} \operatorname{Sin} \chi \operatorname{Cos} \chi \operatorname{Sin}^{2} \theta X^{3}=0  \tag{17}\\
\frac{1}{2} \operatorname{Cot} \chi X^{1}+X_{, 2}^{2}=\alpha  \tag{18}\\
\operatorname{Sin}^{2} \theta X^{3}, 2+X_{, 3}^{2}+\operatorname{Sin} \theta \operatorname{Cos} \theta X^{3}=0  \tag{19}\\
\frac{1}{2} \operatorname{Cot} \chi^{2} X^{1}+X_{, 3}^{3}=\alpha \tag{20}
\end{gather*}
$$

Now integrating equation (12) we get

$$
\begin{equation*}
X^{0}=\alpha t+A^{1}(\chi, \theta, \phi), \quad X^{1}=\alpha \chi+A^{2}(t, \theta, \phi) \tag{21}
\end{equation*}
$$

Now using equation (21) in equations (18) and (20), we get
$X^{2}=\alpha \theta-\frac{\alpha}{2} \theta \chi \cot \chi-\frac{1}{2} \cot \chi \int A^{2}(t, \theta, \phi) d \theta+A^{3}(t, \chi, \phi)$,
$X^{3}=\alpha \phi-\frac{\alpha}{2} \phi \chi \cot \chi-\frac{1}{2} \cot \chi \int A^{2}(t, \theta, \phi) d \phi+A^{4}(t, \chi, \theta)$,
In the above equations (21), (22) and (23) the functions $A^{1}(\chi, \theta, \phi), A^{2}(t, \theta, \phi), A^{3}(t, \chi, \phi)$ and $A^{4}(t, \chi, \theta)$ are functions of integration which are to be determined. First we will find teleparallel Killing vector fields.

### 3.1. Teleparallel Killing Vector Fields

For obtaining teleparallel Killing vector fields we will solve equations (12)-(20) with the help of equations (21), (22) and (23) and substituting $\alpha=0$. Equations (21), (22) and (23) now takes the form:

$$
\begin{align*}
X^{0} & =A^{1}(\chi, \theta, \phi), \\
X^{2} & =-\frac{1}{2} \cot \chi \int A^{2}(t, \theta, \phi) d \theta+A^{3}(t, \chi, \phi)  \tag{24}\\
X^{3} & =-\frac{1}{2} \cot \chi \int A^{2}(t, \theta, \phi)
\end{align*}
$$

Now using (24) in equation (13) and solve after differentiating the resulting equation with respect to $\chi$,
we obtain $A^{1}(\chi, \theta, \phi)=\chi K^{1}(\theta, \phi)+K^{2}(\theta, \phi) \quad$ and $A^{2}(t, \theta, \phi)=\frac{1}{\Phi} K^{1}(\theta, \phi) \int \frac{1}{\Phi(t)} d t+\frac{1}{\Phi(t)} K^{3}(\theta, \phi)$. To avoid lengthy details it suffices to note that we shall follow the same lines and use the system of equations (24) in the remaining Killing equations and reach to a solution for equations (12)-(20), which is listed below

$$
\begin{align*}
& X^{0}=c_{0}, \quad X^{1}=0 \\
& X^{2}=\frac{1}{\Phi(t) \sqrt{\sin \chi}}\left(\phi c_{1}+c_{2}\right),  \tag{25}\\
& X^{3}=\frac{1}{\Phi(t) \sqrt{\sin \chi}}\left(\frac{-c_{1}}{\sin \theta}\{\ln (\operatorname{cosc} \theta-\cot \theta)\}+\frac{1}{\sin \theta} c_{3}\right),
\end{align*}
$$

where $c_{0}, c_{1}, c_{2}, c_{3} \in R$. Thus the generators for the teleparallel Killing symmetry on the manifold of FRW spacetime given in (8) are listed below as:

$$
\begin{aligned}
& \frac{\partial}{\partial t}, \frac{1}{\Phi(t) \sqrt{\sin \chi}} \frac{\partial}{\partial \theta}, \frac{\operatorname{cosec} \theta}{\Phi(t) \sqrt{\sin \chi}} \frac{\partial}{\partial \phi} \\
& \frac{\phi}{\Phi(t) \sqrt{\sin \chi}} \frac{\partial}{\partial \theta}-\frac{\operatorname{cosec} \theta}{\Phi(t) \sqrt{\sin \chi}}\{\ln (\operatorname{cosec} \theta-\cot \theta)\} \frac{\partial}{\partial \phi} . \text { In } \\
& \text { anderal relativity the same }
\end{aligned}
$$ spacetime (8) admits six Killing vector fields as [9] $\frac{\partial}{\partial \phi}, \cos \phi \frac{\partial}{\partial \theta}-\cot \theta \sin \phi \frac{\partial}{\partial \phi}$, $\cos \theta \frac{\partial}{\partial \chi}-\cot \theta \sin \theta \frac{\partial}{\partial \theta},\left(\sin \theta \frac{\partial}{\partial \chi}+\cot \chi \cos \theta \frac{\partial}{\partial \theta}\right) \cos \phi-\cot \chi \operatorname{cosec} \theta \sin \phi \frac{\partial}{\partial \phi} \quad$ and $\left(\sin \theta \frac{\partial}{\partial \chi}+\cot \chi \cos \theta \frac{\partial}{\partial \theta}\right) \sin \phi-\cot \chi \operatorname{cosec} \theta \cos \phi \frac{\partial}{\partial \phi}$. It is important to note that, not only the generators of the Killing algebra in teleparallel theory are less than the generators of the Killing algebra in general relativity but also different from one another. Neither the generators in teleparallel theory are obtainable from the generators in general relativity nor can the generators in general relativity be derived from that of teleparallel theory.

### 3.2. Teleparallel Proper Homothetic Vector Fields

Now our aim is to solve equations (12)-(20) completely for $\alpha \neq 0$. Using equations (21)-(23) in equation (13) and solve after differentiation with respect to $t$, we get the unknown functions as
$A^{1}(\chi, \theta, \phi)=\frac{\chi^{2}}{2} \alpha c_{1}+\chi K^{1}(\theta, \phi)+K^{2}(\theta, \phi)$ and $A^{2}(t, \theta, \phi)=\frac{1}{\Phi} K^{1}(\theta, \phi) \int \frac{1}{\Phi(t)} d t+\frac{1}{\Phi(t)} K^{3}(\theta, \phi)$ where as the metric function $\Phi(t)$ becomes $\Phi^{2}(t)=\left(c_{1} t+c_{2}\right), c_{1}, c_{2} \in R\left(c_{1} \neq 0\right)$. When $c_{1}=0$, the spacetime becomes flat. Substitution of all the above information in equations (21)-(23) will lead us to a system of equations
$X^{0}=\alpha t+\frac{\chi^{2}}{2} \alpha c_{1}+\chi K^{1}(\theta, \phi)+K^{2}(\theta, \phi)$,
$X^{1}=\alpha \chi+\frac{1}{\Phi} K^{1}(\theta, \phi) \int \frac{1}{\Phi(t)} d t+\frac{1}{\Phi(t)} K^{3}(\theta, \phi)$,
$X^{2}=\alpha \theta-\frac{\alpha}{2} \theta \chi \cot \chi-\frac{1}{2} \cot \chi\left(\int\left(\frac{1}{\Phi} K^{1}(\theta, \phi) \int \frac{1}{\Phi(t)} d t+\frac{1}{\Phi(t)} K^{3}(\theta, \phi)\right) d \theta\right)+A^{3}(t, \chi, \phi)$,
$X^{3}=\alpha \phi-\frac{\alpha}{2} \phi \chi \cot \chi-\frac{1}{2} \cot \chi\left(\int\left(\frac{1}{\Phi} K^{1}(\theta, \phi) \int \frac{1}{\Phi(t)} d t+\frac{1}{\Phi(t)} K^{3}(\theta, \phi)\right) d \phi\right)+A^{4}(t, \chi, \theta)$,
where $K^{1}(\theta, \phi), K^{2}(\theta, \phi), K^{3}(\theta, \phi)$ are all unknown functions obtained after integration. Using equation (26) into equation (14) and solving, we get an equation of the form $-\frac{1}{2} \sin \chi \cos \chi c_{2}-c_{3}+\Phi \dot{\Phi}\left(\sin ^{2} \chi-\frac{\alpha}{2} \sin \chi \cos \chi\right) \alpha=0$. Which simply means that $c_{2}=0, c_{3}=0$ and $\alpha=0$. This means that no proper homothety exist and the teleparallel homothetic vector fields are just the teleparallel Killing vector fields given in (25).

## 4. SUMMARY AND DISCUSSIONS

In this paper a diagonal tetrad is taken for FRW $K=+1$ model. Teleparallel Lie derivative has been applied to the metric of the above model to obtain teleparallel homothetic equations. It has shown that dimension of the teleparallel Killing vector field is four. The generators of Killing vectors are compared in general relativity and teleparallel theory. The generators of Killing vectors in both the theories are found different. This study also reveals that neither of the generators is obtainable from the other.

One of the main purposes of our study was also to see if the above spacetime admit any extra symmetry other than teleparallel Killing symmetry for the choice of diagonal tetrad. Interestingly, a diagonal tetrad does not allow this spacetime to exhibit teleparallel proper homothetic vector filed.

Our paper also explored that the presence of torsion reduced the number of Killing symmetries and are totally different from Killing vector fields in general relativity.

## REFERENCES

1.Petrov. AZ, Physics, "Einstein spaces" 1969; Pergamon, Oxford University Press.
2.Bokhari.AH, and Qadir. A, "Symmetries of static spherically symmetric spacetimes", Journal of Mathematical Physics 1987; 28: 1019.
3.Feroz. T, Qadir. A,Ziad. M, "The classification of plane symmetric spacetimes by isometries", Journal of Mathematical Physics 2001; 42: 4947.
4.Shabbir.G,Ramzan. M,"Classification of cylindrically symmetric static spacetimes according to their proper homothetic vector fields, Applied Sciences 2007; 9: 148.
5.Shabbir. G, Amur. KB, "Proper homothetic vector fields in Bianchi type I spacetimes", Applied Sciences 2006; 8: 153.
6. Weitzenböck. R,"InvariantenTheorie", Gronningen: Noordhoft 1923.
7.Aldrovandi.R, Pereira. JG, "An introduction to geometrical physics", World Scientific 1995.
8. Sharif. M, Amir. MJ,"Teleparallel Killing vectors of the Einstein Universe", Modern Physics Letters A 2008; 23: 963.
9. Sharif.M,Majeed. B,"Teleparallel Killing vectors of spherically symmetric space-times", Communications in Theoretical Physics 2009; 52: 435.
10.Shabbir. G, Khan. S, Ali. A, "A note on classification of spatially homogeneous rotating space-times according to their teleparallel Killing vector fields in teleparallel theory of gravitation", Communications in Theoretical Physics 2011; 55: 268.
11.Shabbir. G, Khan.S, Amir. MJ,"A note on classification of cylindrically symmetric non static spacetimes according to their teleparallel Killing vector fields in the teleparallel theory of gravitation",Brazilian journal of physics 2011; 41: 184.
12.Shabbir.G,Khan. S,"A note on proper teleparallel homothetic vector fields in non-static plane symmetric Lorentzian manifolds", Romanian Journal of Physics 2012; 57: 571.
13.Shabbir. G, Khan. S, "Classification of teleparallel homothetic vector fields in cylindrically symmetric static space-times in the teleparallel theory of gravitation", Communications in Theoretical Physics 2010; 54: 675.
14. Khan. S,Hussain.T, Khan. GA,"A note on proper teleparallel homothetic motions of well-known spacetime using non diagonal tetrad", Life Science journal 2013; 10 (11s): 87.
15.Nashed. GGL, "Brane world black holes in teleparallel theory equivalent to general relativity and their Killing vectors, energy, momentum and angular momentum", Chinese Physics B 2010; 19: 20401.
16.Shabbir. G, Khan. A, Khan. S, "Teleparallel conformal vector fields in cylindrically symmetric static space-times", International Journal of Theoretical Physics 2013; 52: 1182.


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