J. Appl. Environ. Biol. Sci., 4(7S)428-435, 2014

ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

Simulation of Diffusion Equation in Irregular Domain Using Local Kernel-based Method

Marjan Uddin [1,2], Amjad Ali, Kamran [2], Muhammad Imran [2], and Zeyad Min Ullah [2],

[1]Department of mathematics, University of Leicester, UK
[2]Department of Basic Sciences, University of Engineering and Technology Peshawar

Received: September 1, 2014
Accepted: November 13, 2014

ABSTRACT

In this work, the implementation of local kernel-based method for heat equation is investigated. The method is local and resulted sparse differentiation matrices. Only small dimension of linear systems of equations are to be solved for every center in the domain. This procedure is more efficient and reliable in solving large scaled engineering problems in irregular domain. Three test cases are done, In the first case the square domain is selected, in the second case the L-shaped domain is considered, while in the third case the circular domain is chosen to approximate the solution of the given problem. The accuracy of the method is tested in terms of L_∞ error norm with respect to the density of interpolation nodes, stencil size, multiquadrics kernels.

KEYWORDS: Local kernel-base method, radial kernels, diffusion equation; irregular domain, meshless method.

1 INTRODUCTION

Most problems in engineering sciences may be formulated as coupled partial differential equations. The exact solution of such type equations in many cases are not easy to obtain, particularly in irregular shaped domain. The recent development of such types of kernel-based methods are investigated in the most recent monographs [1-7] and some of their applications are given in [8-13]. The Kernel-based meshless method or Kansa method [14] is the more famous of them. This meshless approach has been extended to symmetric kernel-based method [15] , to the modified collocation method [16] and to the indirect kernel-based method [17]. In contrary to advantages over mesh-depent methods, unfortunately all the mentioned methods fail to perform dealing for problems with large set of collocation points in the domain, because they resulted fully populated differentiation matrices, which are very sensitive to free shape parameters involved in the kernel functions. Sparse interpolation matrices can be obtained by using compactly supported kernel functions. Sarler and Vertnik, developed a local meshless methods which over come all the difficulties of full-populated differentiation matrices [18]. This approach have been applied for a variety of problems, see for example [19-25]. We further extended this approach for solving the diffusion equation in irregular domain.

1.1 Local kernel-based approximation

For a given sample data points of unknown smooth function $u(x)$, $u(x_i), i = 1,2,...,N,$ where the N centers $\{x_1,...,x_N\} \subset \Omega$,where Ω is arbitrary shaped domain and the centers can be chosen anywhere in the domain Ω . The local kernel-based approximation of the function $u(x)$, at each center $x_i \in \Omega$, is obtained in the form

$$s(x_i) = \sum_{j=1}^{n} \alpha_j k(\|x_i - x_j\|), j = 1,2,...,n \in K_i(n) \subset \Omega \qquad (1)$$

*** Corresponding Author:** Marjan Uddin, Department of mathematics, University of Leicester, UK.
Email: mu41@leicester.ac.uk.

where, $\alpha's$ are the expansion coefficients, $r = \|x - x_j\|$, is the Euclidean norm between two centers x and x_j, $\kappa(r)$ is a radial kernel function defined for $r \geq 0$ and $K_i(n)$ is a vector contains the index of center x_i along with the indices of the reaming $n-1$ centers. This set of centers for the indexed set $K_i(n)$ is call a stencil as shown in figure 1. Consequently, we have N number of $n \times n$ small size linear systems of equations, $s^i = B^i \alpha^i, i = 1, 2, ..., N,$ (2)

the entries of the matrix B^i are $b^i_{kj} = k(\|x_k - x_j\|), k, j \in K_i(n)$, the matrix B^i is called the system matrix, we have to solve each system for the unknown coefficients. Next, we approximate the linear differential operator $L_s(x)$, by $Ls(x_i) = \sum_{j=1}^{n} \alpha_j L\kappa(\|x_i - x_j\|), j = 1, 2, ..., n \in K_i(n) \subset \Omega$

(3)we write eq.(3) as the dot product of two vectors, given by

$Ls(x_i) = v^i \circ \alpha^i$ (4)where the $n \times 1$ vector α^i is unknown coefficients, and $1 \times n$ vector v^i have the entries

$v^i = L\kappa(\|x_i - x_j\|), j = 1, 2, ..., n \in K_i(n),$ (5)

using eq.(2), we eliminate the unknown coefficients,

$\alpha^i = (B^i)^{-1} s^i,$ (6)

by inserting the values of α^i from (6) in (4) to get,

$Ls(x_i) = v^i (B^i)^{-1} s^i = w^i s^i$ (7) where,

$w^i = v^i (B^i)^{-1},$ (8)

denote the corresponding weight for the center x_i. Consequently for every centers locations, the kernel-based spatial approximation of the linear differential operator is obtained

$Ls = Ws,$ (9)

Where the $N \times N$ sparse differentiation matrix W having n non-zeros entries, and $N - n$ with zeros entries, where n is stencil size. When kernel-based local approximation has done, then the problem defined by

$u_t = Lu$ (10) reduced to the system of ODEs in the form

$s_t = F(s),$ (11)

In the present case, we have $F(s) = W(s)$.The time integration may be carried out by any ODE solver, e.g. ode113, ode45, which are built-in Matlab. The initial solution vector would be the initial solution u_0. An appropriate ODE solver adaptively select a correct time step size δt, which overcome stiffness of our ODEs system. In the present work, we use MQ radial kernel function, which contains a free scale factor known as a shape parameter, defined as $\kappa(r) = \sqrt{1 + (\varepsilon r)^2}$. The solution accuracy is very sensitive the scale factor ε. We have obtained the correct scale to get maximum accuracy. A variety of criteria are available in the literature. In the present case, we are using the procedure for getting optimal value of shape parameter available in the literature [7]. In this procedure search for ε, when it satisfy the condition $10^{13} < \kappa < 10^{15}$, κ is the condition number of the system matrix B. We then use singular value

decomposition of the system matrix B^i by $[U, M, V = svd(B^i)]$, Here the $N \times N$ matrices U, and V are orthogonal, and the $N \times N$ matrix M is diagonal with N singular values of the system matrix B^i, and $\kappa = \left\| B^i \right\| \left\| (B^i)^{-1} \right\| = \max(M)/\min(M)$ is the condition number of the system matrix B^i.

3. Numerical experiments

Here we implement the kernel-base local meshless method developed above for the diffusion equation

$$w_t = \nabla^2 w, \qquad (12)$$

with the analytic solution

$$w(x,t) = \sin(\pi x)\sin(\pi y)\exp(-2\pi^2 t).$$

3.1 Rectangular domain

We select the rectangular domain $[-1,1]^2$ to approximate the solution of diffusion equation using the kernel-based meshless method derived in the above section. The time integration are carried out by Rk4 method. The step size $\delta t = 0.001$, the interpolation nodes $N = 400$ are selected in our computations. The approximate solution in terms of the L_∞ error norm, the stencil size n, the condition number κ, of the system matrix are shown in table 1. For maximum sparsity, we have to choose stencil size n much smaller than the number of centers in the domain. However the stencil size n is problem dependent and is related to the condition number κ of the interpolation matrix. The stencil size is selected in such a way that the condition number of the interpolation matrix be kept within the specified range as discussed above. The sparsity pattern shows the location of non-zero entries in the differentiation matrices as shown in figure 1. These sparsity pattern show how well is the kernel-based method, which resulted full differentiation matrices. The present method is very well suited to large scaled problems, where the classical global kernel-based method may faces problems for its implementations.

Table 1. Numerical results with different stencil sizes n, when $N = 400$, $t = 0.1, \delta t = 0.001$, in rectangular domain.

n	L_∞	κ c	C.time(s)
6	2.6012e-003	2.3567e+013	0.0100
10	7.0388e-004	6.1889e+013	0.2100

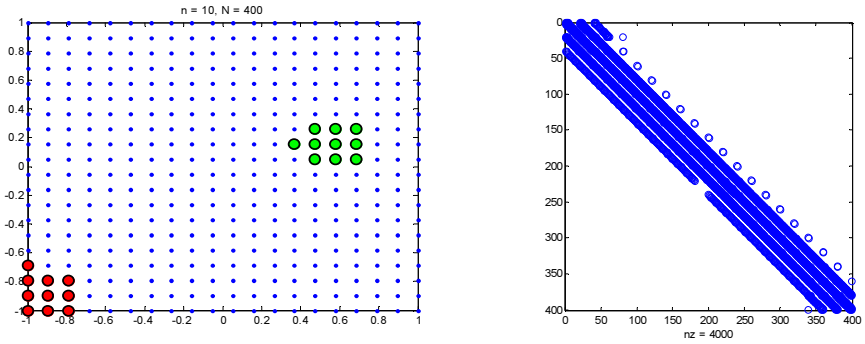

Figure 1. Centers distributions and a stencil corresponding to boundary center (red) and an interior center (green), and the sparsity pattern of the differentiation matrix W, when rectangular domain is used for $N = 400, t = 0.1, \partial t = 0.001.$

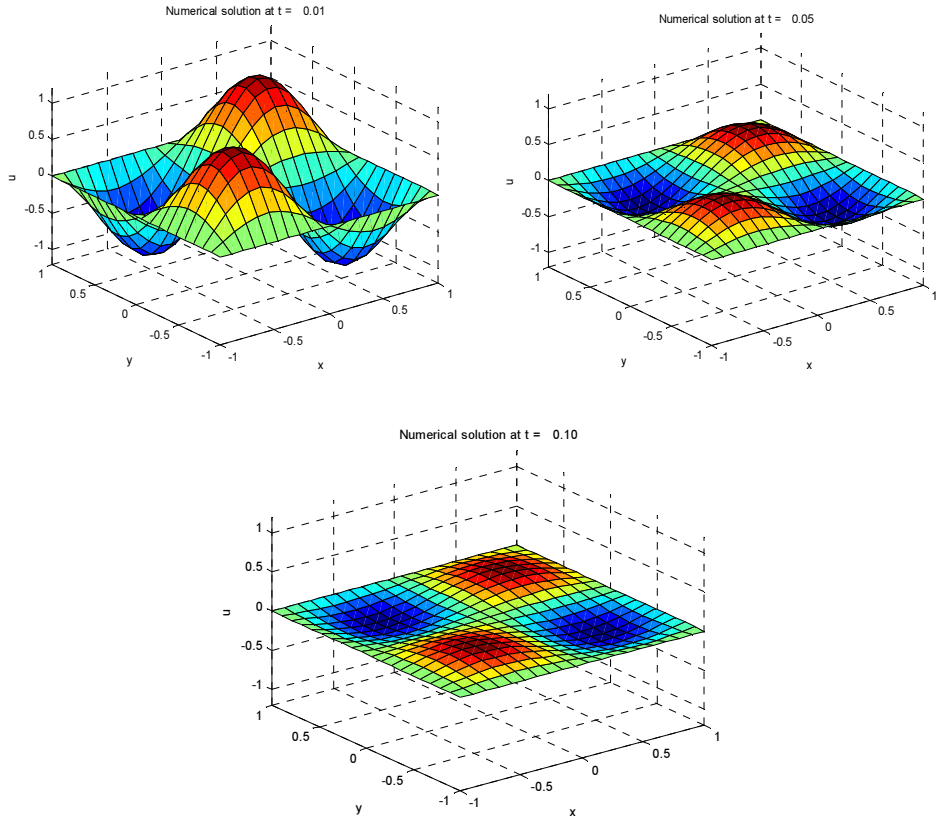

Figure 2. Approximate solution of diffusion equation in the rectangular domain using kernel-based method, when $N = 400, t = 0.1, \partial t = 0.001.$

3.2 L-shaped domain

We apply the local kernel-based method for approximating the L-shaped domain shown in figure 3. The Runge-Kutta method of order four is used to integrate in time the diffusion equation. In this numerical experiment the time step size $\delta t = 0.001$ is selected. The uniformly distributed centers $N = 176$ are selected in the L-shaped domain. The approximate solution in terms of the L_∞ error norm, the stencil size n, the condition number κ of the system matrix is investigated and is shown in table 2. The centers distributions and sparsity of the differentiation matrix W are given in figure 3. The approximate solution of the diffusion equation via local kernel-based method at various time are shown in figure 4. The results again demonstrate the effectiveness of the current method in such a complicated domain for simulating the diffusion equation.

Table 2: Numerical results with different stencil sizes n, when N=176, $t = 0.1, \delta t = 0.001$, in L-shaped domain.

n	L_∞ κ c	C.time(s)
6	2.6012e-0032.3567e+0130.0100	1.337908
10	7.0388e-0046.1889e+0130.2100	1.480991

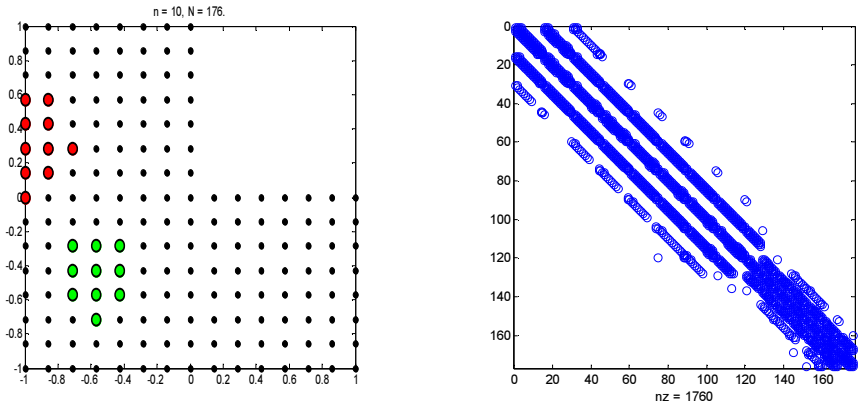

Figure 3. Centers distributions, sparsity pattern of the matrix W, when L-shaped domain is used.

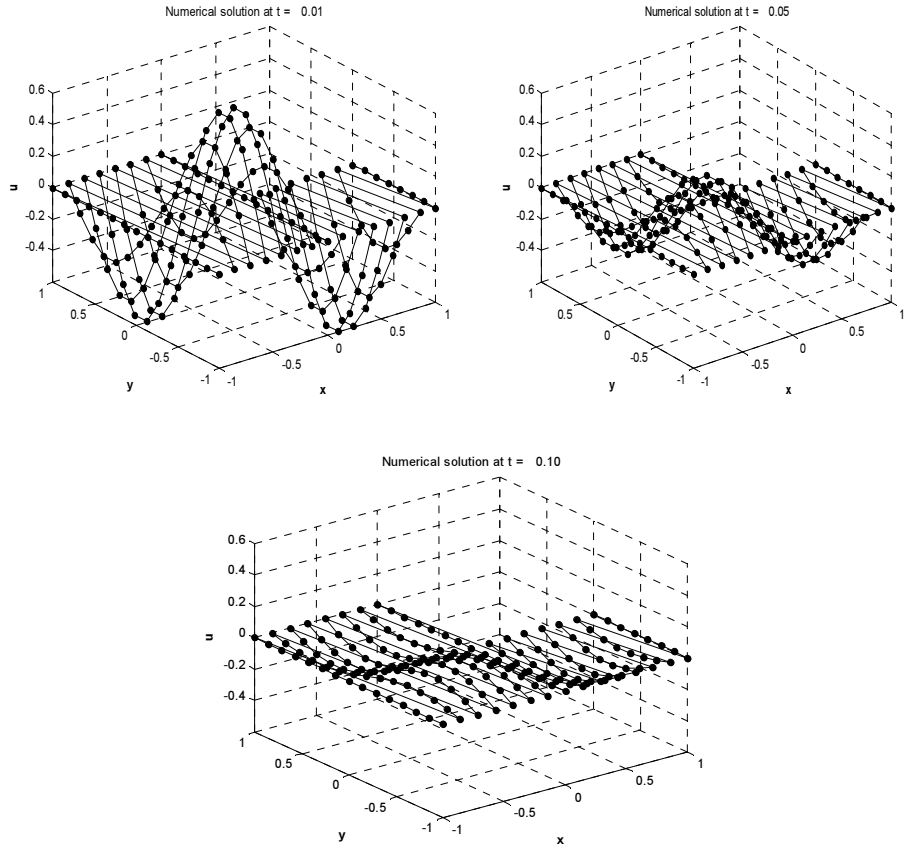

Figure 4. Approximate solution of diffusion equation in the L-shaped domain using kernel-based method, when $N = 176, t = 0.1, \partial t = 0.001$.

3.3 Circular domain

Now we approximate the solution in a circular domain of radius $R = 1$ centered at origin. For time integration once again we used Runge-Kutta method to simulate the diffusion equation with time step

$\delta t = 0.001$. The interpolation nodes $N = 225$ are selected which are uniformly distributed in the circular domain. The results via local kernel-based method are given in table 3 and figures 4-5. Once again the local-kernel based method performed very well in the circular domain.

Table 3: Numerical results with different stencil sizes n, when N=225, $t = 0.1, \delta t = 0.001$, in circular domain.

n	L_∞	κ c	C.time(s)
9	1.1384e-003	8.4252e+0140.0100	2.214279
10	3.8413e-003	2.3019e+0130.0800	1.455471

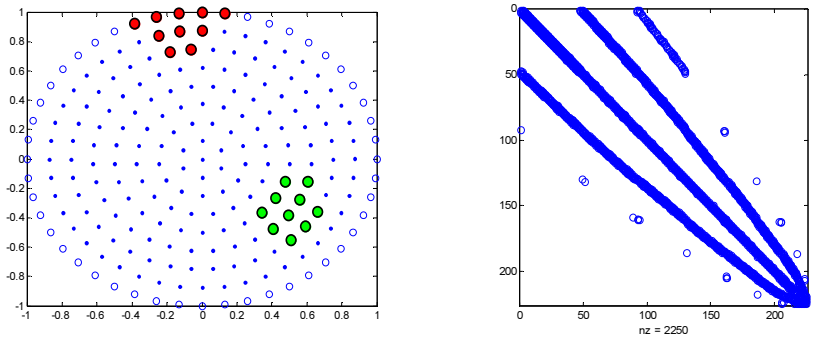

Figure 5. Centers distributions, sparsity pattern of the matrix W, when circular domain is used.

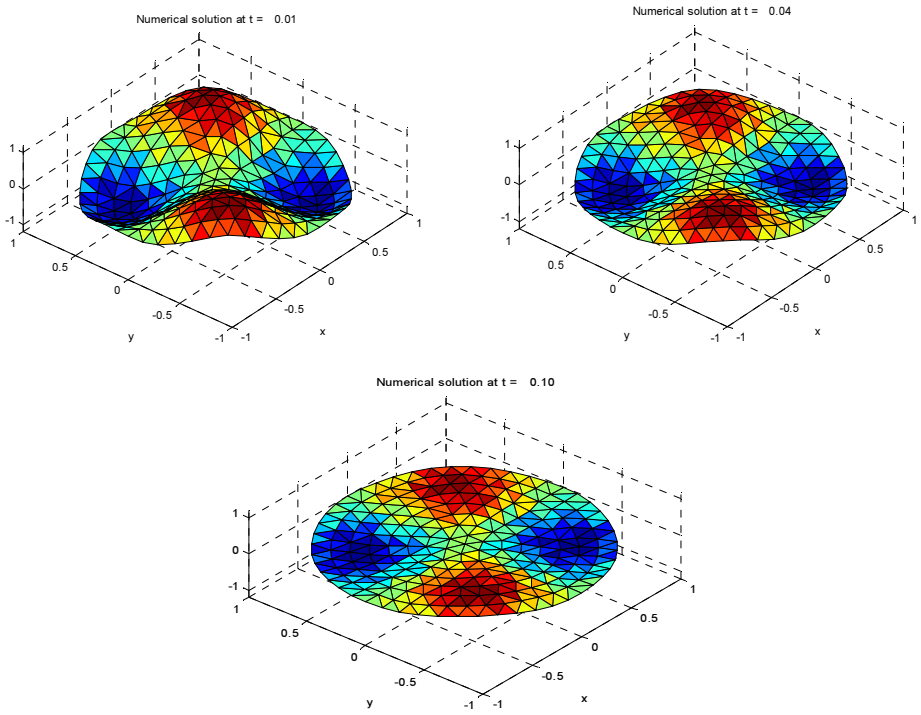

Figure 6. Approximate solution via local kernel-based method in circular domain.

3. Conclusion.

In this work we extend the work of authors in [18] for approximating the solution of diffusion equation in irregular domain. By the use these radial kernels the present method have a great potential for solving many problems in higher dimensions with irregular shaped domain. It is not easy to implement the global kernel-based method for problems with large set of insolation nodes in the domain. However the present local kernel-based method have the capability of solving problems with large data sites in the domain. This procedure has the flexibility to keep the differentiation matrix sparse. This relatively new kernel-based approach is (very) simple meshless formulation for solving a wide range of diffusion problems. The time-marching is done with RK4 method. In the present method the complex-shaped domain can easily incorporated. This meshless procedure appears efficient, it does need full dense systems like the Kansa's global approach. But only small dimension system matrices have to be solved in time step corresponding to each center in the domain. The procedure discussed in this work can be easily be extended to solve other similar types of partial differential equations.

REFERENCES

1. Atluri S. N., and S. Shen, (2002). The Meshless Method. *Tech Science Press, Forsyth.*
2. Liu G. R. (2003). Meshfree methods: moving beyond the finite element method, Boca Raton: CRC Press.
3. Buhmann M. D. (2003) . Radial Basis Functions, *Cambridge University Press.*
4. Sarler B (2004) . Chapter 9: In Advanced Numerical Methods in Heat Transfer, Edited by A. J. Nowak, *Silesian Technical University Press, Gliwice*, 225-247.
5. Wendland H. (2005). Scattered Data Approximation, *Cambridge University Press*.
6. Fasshauer E. G. (2007). Meshfree Approximation Methods with MATLAB, *World Scientific Publishers, Singapore*.
7. Sarra S. A, (2009). Multiquadric radial basis function approximation methods for the numerical solution of partial differential equations, *Adv. Comput. Mech.*
8. Uddin M, H. Ali, (2013). The time marching meshless method of fundamental solutions for wave equations, *VFAST Tran ctions on Mathematics,* 2(1) 1-7.
9. Uddin M., S. Haq, A. Ali, (2014). RBFs meshless method of lines for the numerical solution of nonlinear sine-Gordon equation, *Walailak Journal of Science and Technology (WJST*), 11 (4) 349-360.
10. Hussain A., S. Haq, M. Uddin, (2013). Numerical solution of Klein- Gordon and sine-Gordon equations by meshless method of lines, *Engineering Analysis with Boundary Elements*, 37 (11) 1351-1366.
11. Uddin M., S. Haq, (2013). RBFs meshless method of lines for the numerical solution of nonlinear sine-Gordon equation, *Walailak Journal of Science and Technology (WJST*), 11 (4) 349-360.
12. Uddin M., S. Haq, (2013). On the numerical solution of generalized nonlinear Schrodinger equation using RBFs, *Miskock Mathematical Notes*, 14 (3) 1067-1084.
13. Haq S., S. U. Islam, M. Uddin, (2009). Numerical solution of nonlinear Schrodinger equations by collocation method using radial basis functions, *CMES: Computer Modelling in Engineering and Sciences*, 44 (2) 115-136.

14. Kansa E. J. (1990). Multiquadrics A scattered data approximation scheme with applications to computational fluid dynamics II. Solutions to parabolic, hyperbolic and elliptic partial differential equations, *Computers Math. Applic*. 19, 147-161.

15. G. E. Fasshauer G. E, (1997). Solving partial differential equations by collocation with radial basis functions, *In Surface Fitting and Multiresolution Methods*, (Edited by A. L. Mehaute, C. Rabut and L. L. Schumaker), 131-138.

16. Power H, and W. A. Barraco, (2002). Comparison analysis between unsymmetric and symmetric RBFCMs for the numerical solution of PDEs, *Computers Math. Applic.* 43 (3-5), 551-583.

17. Chen W, (2002). New RBF collocation schemes and kernel RBFs with applications, *Lecture Notes in Computational Science and Engineering*, 26, 75-86.

18. Sarler B, and R. Vertnik, (2006). Meshfree Explicit Local Radial Basis Function Collocation Method for Diffusion Problems, *Computers and Mathematics with Applications*, 51, 1269-1282.

19. Sarra S. A, (2012). A local radial basis function method for advection-diffusion-reaction equations on complexly shaped domains, *Applied Mathematics and Computation*, 218 (190) 9853- 9865.

20. Vertnik R, and B. Sarler, (2006). Meshless local radial basis function collocation method for convective-diffusive solid-liquid phase change problems, *International Journal of Numerical Methods for Heat and Fluid Flow*, 16 (5) 617-640.

21. Yao G, S. U. Islam and B. Sarler, (2010). A comparative study of global and local meshless methods for diffusion-reaction equation, CMES: Computer Modeling in Engineering and Sciences, 59 (2) , 127-154.

22. Yao G, C. S. Chen, J. Kolibal, (2011). A localized approach for the method of approximate particular solutions, *Comput. Math. App*., 61, 2376-87.

23. Ming L, W. Chen, C. S. Chen, (2013). The localized RBFs collocation methods for solving high dimensional PDEs, *Engineering Analysis with Boundary Elements*, 37 1300-1304.

24. Uddin M, M. Imran, (2013). A Local RBF approximation method for time-dependent partial differential equations, *International Conference on Modeling and Simulation ICOMS-2013. Air University, Islamabad, Pakistan*, 75-78.

25. Haider, N. (2013). Numerical solutions of SVEIRS model by meshless and finite difference methods. *VFAST Transactions on Mathematics*, *2*(2), 13-18.