J. Appl. Environ. Biol. Sci., 4(2)299-305, 2014
ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

U-Equivalence Spaces

Farshad Omidi*, Mohammad Reza Molaei

Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran
Received: December 1 2013
Accepted: December 21 2013

ABSTRACT

In this paper the notion of $\mathcal{U}$-equivalence space is introduced. It is proved that the topology induced by a $\mathcal{U}$-equivalence space is regular. $\mathcal{U}$-equivalent continuous functions and $\mathcal{U}$-equivalent open functions are studied. Finally, the quotient $\mathcal{U}$-equivalence spaces are introduced and discussed.
KEYWORDS: U-Equivalence; space; topology; function

INTRODUCTION

Uniform spaces are somewhere the midway points between metric spaces on one hand and abstract topological spaces on the other hand.
There are however a few aspects of metric spaces that are lost in general topological spaces. For example, since the notion of nearness is not defined for a general topological space, we cannot define the notion of uniform continuity in abstract topological spaces. The same can be said about the other notions such as total boundedness. A uniform space, which is due to A. Weil [7] is a mathematical construction in which such 'uniform' concepts are still available.
In this paper we introduce a new construction, namely, $\mathcal{U}$-equivalence space that is almost like a uniform space [4, 5]. We will show that the topological space induced by a $\mathcal{U}$-equivalence space, is a regular topological space. In the theory of $\mathcal{U}$-equivalence spaces, the structure-preserving functions, in the inverse image sense, are $\mathcal{U}$-equivalent continuous functions which are considered in section 3. Also, there is another way of forming a category where the objects are $\mathcal{U}$-equivalence spaces and the morphisms are structure-preserving functions in the direct image sense. We refer to these functions as the $\mathcal{U}$-equivalently open functions (see [4, 5]).
The notion of quotient uniform space was introduced by I.M. James [4]. We introduce and discuss a suitable notion for the quotient $\mathcal{U}$-equivalence space in section 4. In particular we explore several properties of such spaces.

BASIC NOTIONS

Let us begin this section with the definition of the $\mathcal{U}$-equivalence class on a set.

Definition. A $\mathcal{U}$-equivalence class on a set X is a non-empty collection $\mathcal{U}_e$ of equivalence relations on X such that $\mathcal{U}_e$ is closed under finite intersections.
A simple example of a $\mathcal{U}$-equivalence class on a set X, is the collection of all equivalence relations on X which is called discrete $\mathcal{U}$ -equivalence class.

Theorem. The collection $\gamma_e = \{\mathrm{U}(a) \mid a \in \mathrm{X}, \mathrm{U} \in \mathcal{U}_e\}$, where
$\mathrm{U}(a) = \{x \in \mathrm{X} \mid (a, x) \in \mathrm{U}\}$ forms a base for a topology on X.
The topology generated by this base, is called $\mathcal{U}$-equivalence topology and denoted by τ_e.
Corollary. Let $\mathrm{G} \in \tau_e$ and $x \in \mathrm{G}$. Then there exists $\mathrm{U} \in \mathcal{U}_e$ such that $x \in \mathrm{U}(x) \subseteq \mathrm{G}$. Hence the collection $\{\mathrm{U}(a) \mid \mathrm{U} \in \mathcal{U}_e\}$ forms a local base [1,3] at a.
Proof. By theorem 2.2, there exists $\mathrm{U}(a)$ such that $x \in \mathrm{U}(a) \subseteq \mathrm{G}$.
Since $x \in \mathrm{U}(a)$ and U is an equivalence relation on X, then $\mathrm{U}(x) = \mathrm{U}(a)$. Hence $x \in \mathrm{U}(x) \subseteq \mathrm{G}$ as asserted. ■
Proposition. Let $(\mathrm{X},\mathcal{U}_e)$ be a $\mathcal{U}$-equivalence space. Then the following statements are equivalent:
1. The topological space (X, τ_e) is a Hausdorff topological space.
2. The intersection of all members of $\mathcal{U}_e$ coincides with Δ_{x}.

Proof. Suppose (1) holds. Since Δ_{x} is contained in any member of $\mathcal{U}_e$, then
$\Delta_{\mathrm{x}} \subseteq \cap\, \mathrm{U}$ as U ranges over all members of $\mathcal{U}_e$.
For the other way inclusion, assume (x, y) belongs to each U, we will show that

Corresponding author: Farshad Omidi, Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran

$x = y$. If this is not so, then since X is Hausdorff, there exists $U \in \mathcal{U}_e$ and $V \in \mathcal{U}_e$ such that $U(x) \cap V(y) = \Phi$. If $W = U \cap V$, then $W \in \mathcal{U}_e$ and $W(x) \cap W(y) = \Phi$, whence $(x, y) \notin W$ that is a contradiction with assumption.
(2)⇒(1). Assume x, y are distinct members of X. Then by (2), there exists $U \in \mathcal{U}_e$ such that $(x, y) \notin U$. Hence $U(x) \cap U(y) = \Phi$. So the topological space (X, τ_e) is a Hausdorff topological space. ■

Definition. Let A, B be subsets of a $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$ We say that A and B are $\mathcal{U}$-adjacent if for each $U \in \mathcal{U}_e$ there exists $a \in A$ and $b \in B$ such that
$(a, b) \in U$. In particular if $x_o \in X$ and $A \subseteq X$, x_o is adjacent to A if and only if for each $U \in \mathcal{U}_e$, there exists $a \in A$ such that $(x_o, a) \in U$.

Proposition. Let $(X, \mathcal{U}_e)$ be a $\mathcal{U}$ -equivalence space, $x_o \in X$ and let $A \subseteq X$. Then x_o is $\mathcal{U}$-adjacent to A if and only if $x_o \in \bar{A}$ where $\bar{A}$ is the closure of A with respect to τ_e.

Proof. Suppose x_o is adjacent to A and G is a neighbourhood of x_o. By corollary 2.3, there exists $U \in \mathcal{U}_e$ such that $U(x_o) \subseteq G$.
Since x_o is adjacent to A, then there exists $a \in A$ such that $(x_o, a) \in U$.
Hence $U(x_o) \cap A \neq \Phi$. This implies $G \cap A \neq \Phi$. So $x_o \in \bar{A}$.
Conversely let $x_o \in \bar{A}$ and let $U \in \mathcal{U}_e$. Since $U(x_o)$ is a neighbourhood of x_o, then $U(x_o) \cap A \neq \Phi$. Let $a \in U(x_o) \cap A$. Then $a \in A$ and $(x_o, a) \in U$ as required. ■

Theorem. Every $\mathcal{U}$-equivalence space is a regular topological space.

Proof. We first show that the set $U(A) = \{x \in X \mid (a, x) \in U \text{ for some } a \in A\}$ is open and $A \subseteq U(A)$, where $U \in \mathcal{U}_e$ and $A \subseteq X$.
Let $x \in U(A)$. Then there exists $a \in A$ such that $(a, x) \in U$. We claim $U(x) \subseteq U(A)$. If $z \in U(x)$, then $(x, z) \in U$. Since U is transitive, then $(a, z) \in U$ and it follows that
$z \in U(A)$. So $U(A)$ is open. Obviously, $A \subseteq U(A)$.
Now suppose $x_o \in X$ and A is a closed subset of X not containing x_o. Then there exists $G \in \tau_e$ such that $x_o \in G$ and $G \cap A = \Phi$. By proposition 2.3, there exists
$U \in \tau_e$ such that $U(x_o) \subseteq G$. Hence, $U(x_o) \cap A = \Phi$. We claim $U(x_o) \cap U(A) = \Phi$.
If this is not so, then $U(x_o) \cap U(A) \neq \Phi$. If $z \in U(x_o) \cap U(A)$, then $(x_o, z) \in U$ and there exists $a_1 \in A$ such that $(a_1, z) \in U$.
Hence $(x_o, a_1) \in U$ or $a_1 \in U(x_o) \cap A$ contradicts that $U(x_o) \cap A = \Phi$. This implies the result. ■
The following corollary follows from theoren 2.7 and proposition 2.4.

Corollary. Let $(X, \mathcal{U}_e)$ be a $\mathcal{U}$ -equivalence space so that $\cap\{U : U \in \mathcal{U}_e\} = \Delta_X$ · Then (X, τ_e) is a T_3-space [1].

Theorem. Let $(X, \mathcal{U}_e)$ be a $\mathcal{U}$-equivalence space. Then the topological space
(X, τ_e) is connected if and only if it admits the trivial $\mathcal{U}$-equivalence class $\{X^2\}$.

Proof. First suppose X admits the trivial equivalence class, i.e. $\mathcal{U}_e = \{X^2\}$. We show that X is connected. To see this, Let G be open and closed in X (with respect to τ_e) and let $G \neq \Phi$. We have to show that $G = X$. Let $x \in G$. By proposition 2.3, there exists $U \in \mathcal{U}_e$ such that $U(x) \subseteq G$. Since $\mathcal{U}_e$ is trivial, then $U(x) = X$. So $G = X$.
So the empty set and the whole set are the only sets in X which are both open and closed and hence X is connected.
Conversely, assume (X, τ_e) is connected and $U \in \mathcal{U}_e$. We have to show that $U = X^2$. Let $x_o \in X$. From the definition of τ_e, we see immediately that $U(x_o)$ is open. Also we show that $U(x_o)$ is closed. To do this, it is sufficient to show that $\bar{U}(x_o) \subseteq U(x_o)$, where $\bar{U}(x_o)$ is the closure of $U(x_o)$ with respect to τ_e [1].
If $z \in \bar{U}(x_o)$, then $U(z) \cap U(x_o) \neq \Phi$. Consequently, $(x_o, z) \in U$ or, $z \in U(x_o)$. So $U(x_o)$ is also closed. So $U(x_o) = \Phi$ or $U(x_o) = X$. Since $x_o \in U(x_o)$, then $U(x_o) = X$. So $U(x) = X$ for all $x \in X$. Consequently, $U = X^2$. Hence $\mathcal{U}_e = \{X^2\}$. ■

Proposition. Let $(X, \mathcal{U}_e)$ be a $\mathcal{U}$-equivalence space and let $A \subseteq X$. Then:

a) $\bar{A} = \cap\{U : U \in \mathcal{U}_e\}$, where $\bar{A}$ is the closure of A with respect to τ_e.
b) U(A) is closed and open in X.

Proof. a) Let $x \in \bar{A}$ and let $U \in \mathcal{U}_e$. Then $A \cap U(x) \neq \Phi$.

If $a \in A \cap U(x)$, then $(a, x) \in U$ and henee, $x \in U(a)$. So $\bar{A} \subseteq \cap\{U : U \in \mathcal{U}_e\}$.

For the other way inclusion, Let $x \in \cap\{U : U \in \mathcal{U}_e\}$ and let G be a neighbourhood of x_o. By corollary 2.3, there exists $U \in \mathcal{U}_e$ such that $U(x) \subseteq G$.

Since $x \in U(A)$, then there exists $a \in A$ such that $(a, x) \in U$. Thus $G \cap A \neq \Phi$. Hence $x \in \bar{A}$. This shows that $\cap\{U : U \in \mathcal{U}_e\} \subseteq \bar{A}$.
b) Evidently, U(A) is open.

On the other hand, by using (a), $\bar{U}(A) = \cap V(U(A) \subseteq U(V(A)) \subseteq U(A)$. The last statement is true, because U is an equivalence relation on X. So, U(A) is closed. ■
The following corollary is easily obtained from part (a) of proposition 2.10.

Corollary. Let $(X, \mathcal{U}_e)$ be a $\mathcal{U}$ -equivalence space. A subset A of X is dense in X (w.r.t τ_e) If and only if $U(A) = X$ for every $U \in \mathcal{U}_e$.

CONTINUITY
In the theory of $\mathcal{U}$-equivalence spaces the structure-preserving functions, in the inverse-image sense, are the $\mathcal{U}$-equivalently continuous functions, defined as follows.
Definition. Let $(X, \mathcal{U}_e)$, (Y, ϑ_e) be $\mathcal{U}$-equivalence spaces, and let $f : X \rightarrow Y$ be a function. f is said to be $\mathcal{U}$-equivalently continuous if $f_2^{-1}(V) \in \mathcal{U}_e$ for each $V \in \vartheta_e$, where $f_2^{-1}(V) = \{(x, y) \in X^2 \mid (f(x), f(y)) \in V\}$.

Clearly the identity function on any $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$ is $\mathcal{U}$-equivalently continuous.

Definition. A $\mathcal{U}$-equivalence class $\mathcal{U}_e$ is said to be saturated if $U \in \mathcal{U}_e$ and $U \subseteq V$, where V is an equivalence relation on X, then $V \in \mathcal{U}_e$. Also, $\mathcal{U}_e$ is said to be rich if $X^2 \in \mathcal{U}_e$.

Proposition. Let $(X, \mathcal{U}_e)$ and (Y, ϑ_e) be two $\mathcal{U}$ -spaces and let $\mathcal{U}_e$ be saturated.

Then a function $f : X \rightarrow Y$ is $\mathcal{U}$-equivalently continuous, if for each $V \in \vartheta_e$
there exists $U \in \mathcal{U}_e$ such that $f_2(U) \subseteq V$.
Proof. The 'only if' part of the proposition is a simple consequence of definition 3.1. To prove the 'if' part, let $V \in \vartheta_e$. We will show that $f_2^{-1}(V) \in \mathcal{U}_e$. If $U \in \mathcal{U}_e$ and $f_2(U) \subseteq V$, then $U \subseteq f_2^{-1}(V)$. Since V is an equivalence relation on Y, then $f_2^{-1}(V)$ is an equivalence relation on X. Now since $\mathcal{U}_e$ is saturated, $f_2^{-1}(V) \in \mathcal{U}_e$ as asserted. ■

Proposition. Let $(X, \mathcal{U}_e)$, (Y, ϑ_e) be $\mathcal{U}$-equivalence spaces and let $f : X \rightarrow Y$ be $\mathcal{U}$-equivalently continuous function. Then f is continuous when regarded as a function from topological space X in to topological space Y.
Definition. The $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$ is said to be $\mathcal{U}$-connected if for each $U \in \mathcal{U}_e$, $X^2 = \bigcup_{n=1}^{\infty} U^n$ where $U^n =$ U o U o…o U (n-times).

For example, the discrete $\mathcal{U}$-equivalence space X is never $\mathcal{U}$-connceted provided that the underlying set has at least two points. On the other hand, the trivial $\mathcal{U}$-equivalence space is always $\mathcal{U}$-connected.
Defintion. The $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$ is totally bounded if for each $U \in \mathcal{U}_e$, there exist $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} U(x_i)$. For example, the trivial $\mathcal{U}$-equivalence space is always totally bounded.

Definition. Let $(X, \mathcal{U}_e)$, (Y, ϑ_e) be $\mathcal{U}$-equivalently spaces and $f : X \rightarrow Y$ be a function. f is said to be $\mathcal{U}$-equivalently open if for each $U \in \mathcal{U}_e$, there exists $V \in \vartheta_e$ such that $V(f(x)) \subseteq f(U(x))$ for all $x \in X$.

Proposition. Let $f : X \rightarrow Y$ be a $\mathcal{U}$-equivalently continuous surjection, where $(X, \mathcal{U}_e)$ and (Y, ϑ_e) are $\mathcal{U}$-equivalence spaces. Moreover let X be totally bounded. Then so is Y.

Proof. Let $V \in \vartheta_e$. We claim that there exist $y_1, y_2, \ldots, y_n \in Y$ so that $Y = \bigcup_{i=1}^{n} V(y_i)$.

Suppose $U = f_2^{-1}(V)$, then $U \in \mathcal{U}_e$, because f is $\mathcal{U}$-equivalently continuous.
Since X is totally bounded, then there exist $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} U(x_i)$.

If $y_i = f(x_i)$, then we will show that $Y = \bigcup_{i=1}^{n} V(y_i)$. Let $y \in Y$. Since f is surjective, then $y = f(x)$ for some $x \in X$.

For $i = 1, 2, \ldots, n$, let $(x_i, x) \in U = f_2^{-1}(V)$, then $(f(x_i), f(x)) \in V$. Hence $Y = \bigcup_{i=1}^{n} V(y_i)$. as asserted. ■

Proposition. Let $f : X \to Y$ be a $\mathcal{U}$-equivalently continuous surjection, where $(X, \mathcal{U}_e)$ and (Y, ϑ_e) are $\mathcal{U}$-spaces. If X is $\mathcal{U}$-connected, then so is Y.

Proof. Let $V \in \vartheta_e$. Since f is surjection, then so is f_2. Since f is $\mathcal{U}$-equivalently continuous, then $U = f_2^{-1}(V) \in \mathcal{U}_e$. So, $Y^2 = f_2(X^2) = f_2(\bigcup_{n=1}^{\infty} U^n) = \bigcup_{n=1}^{\infty} f_2(U^n) = \bigcup_{n=1}^{\infty} V^n$

Hence, Y is $\mathcal{U}$-connected and the proof is now complete. ■

Poroposition 3.10. Let $(X, \mathcal{U}_e)$, (Y, ϑ_e) and $(Z, \mathcal{w}_e)$ be $\mathcal{U}$-equivalence spaces and $f : X \to Y$ be a $\mathcal{U}$-equivalently continuous surjection and let $g : Y \to Z$ be a function.
If $g \circ f : X \to Z$ is $\mathcal{U}$-equivalently, open then so is g.

Proof. let $V \in \vartheta_e$ and $U = f_2^{-1}(V)$. Since f is $\mathcal{U}$-equivalently continuous, then $U \in \mathcal{U}_e$. Moreover, since f is $\mathcal{U}$-equivalently open, then there exists $W \in \mathcal{w}_e$ such that
$W((g \circ f)(x)) \subseteq (g \circ f)(U(x))$ for all $x \in X$. We clain that $W(g(y)) \subseteq g(V(y))$ for all $y \in Y$. To see this, let $y \in Y$ and $z \in W(g(y))$. Since f is surjection, then $y = f(x)$ for some $x \in X$. So $W(g(y)) \subseteq (g \circ f)(U(x))$ (I).
Hence by (I), there exists $x_1 \in X$ such that $(x, x_1) \in U$, $z = g(f(x_1))$. Let $t = f(x_1)$. Then $z = g(t)$, $(y, t) \in V$ i.e. $z \in g(V(y))$ as required. ■

Let us present another classification of saturated $\mathcal{U}$-connected spaces as follows.

Theorem. In a saturated $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$ the following statements are equivalent:

1) X is $\mathcal{U}$-connected

2) for each discrete space D, every $\mathcal{U}$-equivalently continuous function $\lambda : X \to D$ is constant.

Proof. (1) $\to$ (2). Given a $\mathcal{U}$-equivalently continuous function $\lambda : X \to D$
Where D is discrete i.e. $U_D = \{V \subseteq D^2 \mid V$ is an equivalence relation on D$\}$.
Consider the pre-image $U = \lambda_2^{-1}(\Delta_D)$ of the diagonal Δ_D of D. Then $U \in \mathcal{U}_e$, and $U^n = U$ for all n, because $\Delta_D^n = \Delta_D$. Since X is $\mathcal{U}$-connected then $X^2 = \bigcup_{n=1}^{\infty} U^n = U$.

On the other hand, $U = \{(x_1, x_2) \in X^2 \mid \lambda(x_1) = \lambda(x_2)\}$. Hence λ is constant.
(2) $\to$ (1). Suppose that X is not $\mathcal{U}$-connected.
Then there exists $U \in \mathcal{U}_e$ and $x_o, y_o \in X$ such that $(x_o, y_o) \notin U^n$ for all n. Taking $D = \{0, 1\}$ equipted with discrete $\mathcal{U}$-equivalence class.
Define $\lambda : X \to D$ by $\lambda(x) = 0$ when $(x_o, x) \in D^i$ for some i and $\lambda(x) = 1$ otherwise. Hence $\lambda(x_o) = 0$ and $\lambda(y_o) = 1$ i.e. λ is not constant. We claim that λ is $\mathcal{U}$-equivalently continuous.
We first show that $U \subseteq \lambda_2^{-1}(\Delta_D)$. If this is not so, then there exists $(x_1, x_2) \in U$, $\lambda(x_1) \neq \lambda(x_2)$. Assume that $\lambda(x_1) = 1$, $\lambda(x_2) = 0$. Hence there exists $i \in N$, $(x_o, x_2) \in U^i$. Consequently, $(x_o, x_1) \in U^{i+1}$ contradicting that $\lambda(x_1) = 1$. Hence $U \subseteq \lambda_2^{-1}(\Delta_D)$. So for each $V \in \mathcal{U}_D$, $\lambda_2^{-1}(V) \supseteq \lambda_2^{-1}(\Delta_D) \supseteq U$. Whence $\lambda_2^{-1}(V) \in \mathcal{U}_e$ because $\mathcal{U}_e$ is saturated.

Hence λ is $\mathcal{U}$-equivalently continuous function while it is not constant, that is a contradiction. This proves that X is $\mathcal{U}$-connected. ■

We omit the straightforward proof of the following proposition.

Proposition. Let (X, $\mathcal{U}_e$), (Y, ϑ_e) be $\mathcal{U}$-equivalence spaces where ϑ_e is saturated.Then a bijection f : X → Y is $\mathcal{U}$-equivalently open if and only if its inverse is $\mathcal{U}$-equivalently continuous.

Proposition. Let (X, $\mathcal{U}_e$), (Y, ϑ_e) and (Z, w_e) be $\mathcal{U}$-equivalence spaces and f : X → Y be a function and let g : Y → Z be $\mathcal{U}$ -equivalently continuous injection. If g o f is $\mathcal{U}$-equivalently open, then so is f.

Poof. Let U ϵ $\mathcal{U}_e$. Then there exists W ϵ w_e, W(h(x) ⊆ h(U(x)) for all x ϵ X where h = g o f. Since g is $\mathcal{U}$-equivalently continuous, then the pre-image V = g_2^{-1}(W) is a member of ϑ_e. Now it is easy to see V(f(x)) ⊆ f(U(x)) for all x ϵ X. it follows that f is $\mathcal{U}$ -equivalently open as asserted. ■

Proposition. Let f : X → Y be a $\mathcal{U}$-equivalently open function, where X is non-empty, (X, $\mathcal{U}_e$) is rich and (Y, ϑ_e) is $\mathcal{U}$-connected. Then f is surjection.

Proof. Let U = X^2, Then there exists V ϵ ϑ_e such that V(f(x)) ⊆ f(U(x)) for all x ϵ X. consequently, V(f(x)) ⊆ f(X) for all x ϵ X. Hence for each n and each x ϵ X,

V^n(f(x)) ⊆ f(X). Let x_o ϵ X and let y_o = f(x_o). We claim Y = f(X).

To see this, let y ϵ Y, then (y_o, y) ϵ Y^2 = $\bigcup_{n=1}^{\infty} V^n$. Hence, y ϵ V^n(f(x_o)) for some n. Since V^n(f(x_o)) ⊆ f(X), then y ϵ f(X). This proves Y = f(X). ■

Definition. Let f : X → Y be a map where (X, $\mathcal{U}_e$) is a $\mathcal{U}$-equivalence space and Y is a set. We say that f is transverse to X if there exists U ϵ $\mathcal{U}_e$ such that

U ∩ f_2^{-1}(Δ_Y) = Δ_X. By a local $\mathcal{U}$ -equivalence we mean, a $\mathcal{U}$-equivalently continuous and $\mathcal{U}$-equivalently open function f : X → Y, where (X, $\mathcal{U}_e$) and (Y, ϑ_e) are $\mathcal{U}$-equivalence spaces such that f is transverse to X.

Proposition. Let f : X → Y be a $\mathcal{U}$-equivalently continuous function. Suppose f admits a left inverse g which is local $\mathcal{U}$-equivalence. Then f is $\mathcal{U}$-equivalently open.

Proof. Let U ϵ $\mathcal{U}_e$. Then V_1 = g_2^{-1}(U) ϵ ϑ_e because g is $\mathcal{U}$-equivalently continuous. Since g is transverse to Y, then there exists V_o ϵ ϑ_e such that V_o ∩ g_2^{-1} (Δ_X) = Δ_Y. Let V_2 = (f o g)$_2^{-1}$(V_0). Then since f o g is $\mathcal{U}$-equivalently continuous, V_2 ϵ ϑ_e. Finally let V = V_0 ∩ V_1 ∩ V_2. We claim V(f(x)) ⊆ f(U(x)) for all x ϵ X. suppose y ϵ V(f (x)). Then (x, g(y)) ϵ U. Finally, we have to show that f(g(y)) = y.

Since (g(y), g(y)) ϵ Δ_X, then (f(g(y), y) ϵ g_2^{-1} (Δ_X).

Also, (y, f(x)) ϵ V_0 and (f (x), (f(g(y)) ϵ V_0. Hence, (f(g(y)), y) ϵ V_0 . consequently, (f(g(y)), y) ϵ Δ_Y that means, f(g(y)) = y. ■

Proposition. Let f : X → Y and g : Y → Z be $\mathcal{U}$-equivalently continuous functions, where (X, $\mathcal{U}_e$), (Y, ϑ_e) and (Z, w_e) are $\mathcal{U}$-equivalence spaces g o f is $\mathcal{U}$-equivalently open, f is injective and g is transverse to Y. The g o f is a local $\mathcal{U}$-equivalence.

Proof. Since g is transverse to Y, there exists V ϵ ϑ_e such that V ∩ g_2^{-1}(Δ_Z) = Δ_Y. Let U = f_2^{-1}(V). Then U ϵ $\mathcal{U}_e$. Now we have to show that U ∩ ((g o f)$_2^{-1}$(Δ_z) = Δ_x.

Clearly, Δ_x ⊆ U ∩ ((g o f)$_2^{-1}$(Δ_Z). For the other way inclusion, let (x_1, x_2) ϵ U and g(f(x_1) = g(f(x_2). Then (f(x_1), f(x_2)) ϵ V ∩ g_2^{-1}(Δ_Z) = Δ_Y. So f(x_1) = f(x_2) and since f is injective, x_1 = x_2. Hence U ∩ (g o f)$_2^{-1}$(Δ_Z) = Δ_X. ■

QUOTIENT $\mathcal{U}$ -EQUIVALENCE SPACES

Let (X, $\mathcal{U}_e$) be a $\mathcal{U}$-equivalence space and let $\mathcal{R}$ be an equivalence relation on X.

Also, let $\pi: X \rightarrow X/\mathcal{R}$ is the function defined by $\pi(x) = \mathcal{R}[x]$, where $\mathcal{R}[x] = \{y \in X \mid (x, y) \in \mathcal{R}\}$. The function π is called the natural projection.

Now we ask whether $X/\mathcal{R}$ can inherits a $\mathcal{U}$-equivalence class from X such that makes the natural projcetion π $\mathcal{U}$-equivalently continuous, and if the answer is yes, then we discuss the relationships between these two spaces.

Definition. An equivalence relation $\mathcal{R}$ on a $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$ is compatible with $\mathcal{U}_e$ if for each $U \in \mathcal{U}_e$, $\mathcal{R} \circ U = U$

For example, let X be a non-empty set and $\mathcal{R} = \Delta_X$ Then $\mathcal{R}$ is compatible with $\{X^2\}$.

The following lemma is often useful.

Lemma. Let $\mathcal{R}$ be an equivalence relation on a $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$. Then the following statements are equivalent:

i) $\mathcal{R}$ is compatible with $\mathcal{U}_e$.

ii) For each $U \in \mathcal{U}_e$, $U \circ \mathcal{R} = U$.

iii) For each $U \in \mathcal{U}_e$, $\mathcal{R} \circ U \circ \mathcal{R} = U$.

iv) For each $U \in \mathcal{U}_e$, $U \circ \mathcal{R} \circ U = U$.

Proof. The equivalence of (i) with (ii) is trivial.

Assume (ii) holds and suppose $U \in \mathcal{U}_e$, Then $U \circ \mathcal{R} = U$ and hence $\mathcal{R} \circ U \circ \mathcal{R} = \mathcal{R} \circ U$. Since $U \circ \mathcal{R} = U$, then the equivalence of (i) with (ii) implies $\mathcal{R} \circ U = U$. Hence $\mathcal{R} \circ U \circ \mathcal{R} = U$.The other parts result by straightforward calculations. ■

Theorem. Let $\mathcal{R}$ be a compatible equivalence relation on a $\mathcal{U}$-equivalence space $(X, \mathcal{U}_e)$. Then the images of the members of $\mathcal{U}_e$ under π_2, form a $\mathcal{U}$-equivalence class on $X / \mathcal{R}$. We refe to this class as the quotient $\mathcal{U}$-epuivalence class and to $X / \mathcal{R}$ with this structure, as the quotient $\mathcal{U}$-equivalence space.

We recall that $X / \mathcal{R}$ is the collection of all equivalence classes $\mathcal{R}[X]$, and $\pi_2(x, y) = (\pi(x), \pi(y)) = (\mathcal{R}[x], \mathcal{R}[y])$

Proof. Let $\mathcal{U}_e^{\pi}$ denotes this collection i.e. $\mathcal{U}_e^{\pi} = \{\pi_2(U) \mid U \in \mathcal{U}_e\}$.

We first show each member of $\mathcal{U}_e$ is an equivalence relation on $X / \mathcal{R}$. Let $V = \pi_2(U)$ where $U \in \mathcal{U}_e$ and let $x \in X$.

Then $(\mathcal{R}[x], \mathcal{R}[x]) = \pi_2(x, x)$ and $(x,x) \in \Delta_X \subseteq U$. Hence $\Delta_{X / \mathcal{R}} \subseteq V$ and so V is reflexive. Clearly V is symmetric. Now we show that V is transitive.

Let $(\mathcal{R}[x], \mathcal{R}[y]) \in V$ and let $(\mathcal{R}[y], \mathcal{R}[z]) \in V$. Then $(\mathcal{R}[x], \mathcal{R}[y]) = (\mathcal{R}[t_1], \mathcal{R}[t_2]))$, $(t_1, t_2) \in U$. Also $(\mathcal{R}[y], \mathcal{R}[z]) = (\mathcal{R}(u_1), \mathcal{R}(u_2])$, $(u_1, u_2) \in U$.

Hence $(\mathcal{R}[x], \mathcal{R}[z]) = (\mathcal{R}[t_1], \mathcal{R}[u_2])$. Since $(t_1, t_2) \in U$, $(t_2, u_1) \in \mathcal{R}$ and $(u_1, u_2) \in U$, then $(t_1, u_2) \in U \circ \mathcal{R} \circ U$. Now compatibility of $\mathcal{R}$ with $\mathcal{U}_e$, implies $(t_1, u_2) \in U$. Hence $(\mathcal{R}[x], \mathcal{R}[z]) = \pi_2(t_1, u_2)$, $(t_1, u_2) \in U$.

So $(\mathcal{R}[x], \mathcal{R}[z]) \in \pi_2(U) = V$. Whence V is transitive.

Finally, We show that the intersection of two members of $\mathcal{U}_e^{\pi}$ is a member of $\mathcal{U}_e$. Let $V_1 = \pi_2(U_1)$ and $V_2 = \pi_2(U_2)$, where $U_1, U_2 \in \mathcal{U}_e$, be two members of $\mathcal{U}_e^{\pi}$.

We contend that $V_1 \cap V_2 = \pi_2(U_1 \cap U_2)$ which shows that $V_1 \cap V_2 \in \mathcal{U}_e^{\pi}$

Clearly $\pi_2(U_1 \cap U_2) \subseteq \pi_2(U_1) \cap \pi_2(U_2)$. Now let $(\mathcal{R}[x], \mathcal{R}[y]) \in \pi_2(U_1) \cap \pi_2(U_2)$.

Then $(\mathcal{R}[x], \mathcal{R}[y]) = (\mathcal{R}[t_1], \mathcal{R}[t_2])$, $(t_1, t_2) \in U_1$

$= (\mathcal{R}[u_1], \mathcal{R}[u_2])$, $(u_1, u_2) \in U_2$.

Consequently, $(\mathcal{R}[x], \mathcal{R}[y]) = (\mathcal{R}[t_1], \mathcal{R}[u_2]$. But $(t_1, u_2) \in \mathcal{R} \circ U_2 = U_2$ and,

$(t_1, u_2) \in U_1 \circ \mathcal{R} = U_1$. Hence, $(\mathcal{R}[x], \mathcal{R}[y]) = (\mathcal{R}[t_1], \mathcal{R}[u_2])$, $(t_1, u_2) \in U_1 \cap U_2$. So $(\mathcal{R}[x], \mathcal{R}[y]) \in \pi_2(U_1 \cap U_2)$. Hence, $\pi_2(U_1 \cap U_2) = \pi_2(U_1) \cap \pi_2(U_2)$. ■

Theorem. Let $\mathcal{R}$ be an equivalence relation on X , compatible with $\mathcal{U}_e$ where $(X, \mathcal{U}_e)$ is a $\mathcal{U}$-equivalence space. Then π is $\mathcal{U}$-equivalently continuous and $\mathcal{U}$-equivalently open.

Proof. We first show that π is $\mathcal{U}$-equivalently open. Let $U \in \mathcal{U}_e$ and $V = \pi_2(U)$. Then $V \in \mathcal{U}_e^{\pi}$. We claim that $V(\pi(x)) \subseteq \pi(U(x))$ for all $x \in X$.

Let $x \in X$ and let $\mathcal{R}[t] \in V(\pi(x)) = V(\mathcal{R}[x])$ We will show there exists $u \in X$ such that $\mathcal{R}[t] = \mathcal{R}[u]$ and $(x, u) \in U$. Since $R[t] \in V[\mathcal{R}[x]]$, then there exists $(t_1, t_2) \in U$ such that $(\mathcal{R}[x], \mathcal{R}[t]) = (\mathcal{R}[t_1], \mathcal{R}[t_2])$. Hence $\mathcal{R}[t] = \mathcal{R}[t_2]$ and $(x, t_2) \in \mathcal{R} \circ U = U$. let $u = t_2$. Then $R[t] = R[u]$ and $(x, u) \in U$ as required.

Now we prove that π is $\mathcal{U}$-equivalently continuous. Let $V \in \mathcal{U}_e^{\pi}$. We show that $\pi_2^{-1}(V) \in \mathcal{U}_e$. There exists $U \in \mathcal{U}_e$ such that $V = \pi_2(U)$. On one hand we have $\pi_2^{-1}(V) = \pi_2^{-1}(\pi_2(U)) \supseteq U$. On the other hand, if $(x_1, x_2) \in \pi_2^{-1}(V)$, then $(\mathcal{R}[x_1], \mathcal{R}[x_2]) = (\mathcal{R}[t_1], \mathcal{R}[t_2])$, $(t_1, t_2) \in U$. Hence $(x_1, x_2) \in \mathcal{R} \circ U \circ \mathcal{R} = U$.
So $\pi_2^{-1}(V) \subseteq U$. And hence $\pi_2^{-1}(V) = U \in \mathcal{U}_e$.

Acknowledgment

The authors declare that they have no conflicts of interest in the research.

REFERENCES

[1] J.R. Munkres, Topology-A First Course, Prentice-Hall, New Delhi, 1978.
[2] S. Willard, General Topology, Addison-Wesley, Reading, 1970.
[3] J.L. Kelley, General Topology, Van Nostrand, Princeton, 1955.
[4] I.M. James, Topological and Uniform Speces, Springer, New York, 1987.
[5] W. Page, Topological and Uniform Structures, Wiley, New York, 1978.
[6] R. Fuller, Uniform Continuity and Net Behavior, Annales Societatis Mathematicae Plonae, Comentationes Mathematicae XVI, 1972, 165-167.
[7] A. Weil, Sur les Espaces a Structure Uniforme et sur la Topologie Gnerale, Actualites Sci. Ind. 551, Paris, 1937.
[8] B. Hutton, Uniformities of Fuzzy Topological Spaces, J. Math. Anal. Appl. **58**(1977), 559-571.
[9] D. Zhang, Stratified Hutton Uniform Spaces, Fuzzy Sets and Systems 131, (2002), 337-346.
[10] D.K. Mitra and D. Hazarika, L-locally Uniform Spaces, J. Fuzzy Math.**18**(2002), 505-516.
[11] Turkoglu D, Ozer O and Fisher B, Some Fixed Point Theorems for Set Valued Mapping in Uniform Spaces, Demonstratio Math. **2**(1999), 395-400.
[12] M. Katetov, On Continuity Structures and Spaces of Mappings, Comm. Math. Univ. Carolinae, **6**(1965), 257-278.